Some results on Fixed points for generalized (α,ϕ,ψ) contractive multifunctions in symmetric spaces

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Abstract: Recently Abdelbasset Felhi introduced a class of generalized (α,ϕ,ψ) proximal contraction for non-self maps in semi-metric spaces and gave some results on best proximity points and fixed points. In this paper, we used this generalized (α,ϕ,ψ) contraction for multivalued maps in symmetric spaces for the existence of fixed points and some related results for self maps. We provide sufficient conditions for the existence and uniqueness of fixed points by using the concept of α- admissible.

Key words: Symmetric spaces, multivalued maps, Hausdorff metric, fixed point,α-admissible .

I.Introduction

Semi-metric spaces were considered by many authors like Frechet[7],Menger[12]and Wilson[19] as a generalization of metric spaces.After that some fixed point results for semi-metric spaces have been investigated in [1],[3]-[15].

The contraction is one of the important tool to prove the existence and uniqueness of a fixed point. Banach contraction principle is one of the most fascinating and classical result of the last century in the field of non linear analysis. Following Banach contraction mapping Nadler [16] introduced the concept of multivalued contraction mapping and established that a multivalued contraction mapping possesses a fixed point in a complete metric space. There are so many fixed point theorems for multivalued mappings in metric spaces satisfying contractive type conditions.

On the other hand, Hicks [10], and Hicks and Rhoades [11] started the study of existence of fixed points in symmetric spaces.

Samet, Vetro and Vetro [17] introduced the notion of α-ψ-contractive type mappings and established some fixed point theorems in complete metric spaces. Mohammadi, Rezapour, Shahzad gave some new results for α-ψ-ciric generalized multifunctions.

Abdelbasset Felhi [2] introduced generalized (α,ϕ,ψ) contractions for non-self maps in semi-metric spaces for the existence and uniqueness of best proximity points. The aim of this paper is to establish some fixed points theorems for multivalued mappings using the generalized (α,ϕ,ψ) contractions which was introduced by Abdelbasset Felhi[2].

Our results generalize and improve various known results from fixed point theory.

II. Preliminaries

1. *Definition* [11]: A symmetric of a set X is a non negative real valued function d on $X \times X$ such that (i) d(x,y) = 0 iff x=y (ii) d(x,y) = d(y,x). Let d be a symmetric on a set X and for r>0 and any $x\in X$, let $B(x,r) = \{y\in X: d(x,y)<r\}$. A topology t(d) on X is given by $U \in t(d)$ if and only if for each $x\in U$, $B(x,r)\subset U$ for some r> 0.
2. *Definition [11]:* A symmetric d is said to be semi-metric if for each $x\in X$ and each r> 0, B(x, r) is a neighbourhood of x in the topology t(d) . Note that $\lim_{n\to\infty} d(x_n,x)=0$ if and only if $x_n \to x$ in the topology t(d) or τ_d We need the following two axioms (W.3) and (W.4) given by Wilson[19] in a symmetric space (X,d).

   (W.3) Given $\{x_n\}$, x and y in X , $\lim_{n\to\infty} d(x_n,x)=0$ and $\lim_{n\to\infty} d(x_n,y)=0$ imply x=y.

   (W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x)=0$ and $\lim_{n\to\infty} d(x_n,y_n)=0$ imply that $\lim_{n\to\infty} d(y_n,x)=0$

3. *Proposition [11]:* Let (x, d) be a symmetric space. Then (x, d) is semi-metric space if and only if the following conditions hold:
   (1) (x, τ_d) is first countable
   (2) For any sequence $\{x_n\}$ in X, $d(x_n, x) \to 0$ is equivalent to $x_n \to x$ in the topology τ_d.
4. *Definition[9,11]:* Let (X,d) be a symmetric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is d-cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Furthermore, (X, d) is said to be d-cauchy complete if every d- Cauchy sequence converges to some $x \in X$ in τ_d. It is easy to see that for a semi-metric d, if τ_d is Hausdroff, then (W.3) holds. Let (X, d) be asymmetric space. CB(X) denotes the collection of all closed bounded subsets of X. For any $x \in X$ and A is a non-empty subset of X. $d(x, A) = \inf\{d(x, y) : y \in A)$ and $H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$ .H is known as the Hausdroff metric introduced by d on CB(x)[16].Further, if (x,d) is complete (CB(X),H) is also complete.
5. *Definition:* Let (X,d) be a symmetric space and A is a non empty subset of X.
   1. We say that A is d-closed if and only if $\overline{A}^d = A$ where $\overline{A}^d = \{x \in X : d(x, A) = 0\}$ and $d(x, A) = \inf\{d(x, y) : y \in A\}$
   2. We say that A is d-bounded if and only if $\delta_d(A) = \sup\{d(x, y) : x, y \in A\}$ .For main theorem we need the following lemma.
6. *Lemma[6]*: Let (X,d) be a d-bounded symmetric space. Let A,B$\in$CB(X) and q>1 . For each x∈A, there exists y∈B such that d(x,y)≤qH(A,B).
7. *Definition*: Let (X, d) be a symmetric space. We say that (X,d) satisfies the property (WC) if for all sequences $\{x_n\}$, $\{y_n\}$ in X and all x,y∈X such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(y_n, y) = 0$, one has $d(x, y) \le \lim_{n\to\infty}\inf d(x_n, y_n)$.
8. *Remark:*
   1. If (X,d) be a symmetric space satisfying the property (WC), then it is also satisfying the Fatou property.
   2. Each metric space satisfies the property (WC).
9. *Definition[17]*: Let (X,d) be a symmetric space. Let $T : X \to 2^x$ be a multivalued function then T is said to be α-admissible whenever $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$ for all x,y∈X.
10. *Definition[17]*: Let (X, d) be a symmetric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to 2^x$ is said to be triangular α-admissible if (T1) T is α-admissible, (T2) $\alpha(x, y) \ge 1$ and $\alpha(y, z) \ge 1 \Rightarrow \alpha(x, z) \ge 1,$ for all x,y,z in X .
11. *Definition[18]:*Denote by ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying (ψ1) ψ is non-decreasing; (ψ2) $\lim_{n\to\infty} \psi^n(t) = 0$ for each t>0, where ψ^n is the n-th iterate of ψ.Also denote by ϕ the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying (ϕ1) ϕ is non-decreasing; (ϕ2) $\phi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to 0^+} \phi(x) = 0$
12. *Lemma[18]*: If ψ∈ψ, then $\psi(t) < t$ for all $t > 0$, ψ is continuous at 0 and ψ(0)=0.
13. *Lemma [2]*: Let (X,d) be a symmetric space and ϕ∈Φ. Consider the function $\phi \circ d : X \times X \to [0, \infty)$ defines as follows $\phi \circ d(x, y) = \phi(d(x, y))$ for all x,y∈X. Then (X, $\phi \circ d$) is also a symmetric space. Using the definition of (α,ϕ,ψ) contraction given in [2],we extend it to the multivaued maps.
14. *Definition*: Let (X,d) be a symmetric space, let $T : X \to CB(X), \phi \in \Phi$, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ .Then T is generalized (α,ϕ,ψ) contractive multifunction if α(x,y)≥1 $\Rightarrow \phi\left(H\left(Tx, Ty\right)\right) \le \psi\left(\phi\left(d\left(x, y\right)\right)\right)$ for x,y$\in X$.

    Using the above definition, we extend and generalize the following map.
15. *Definition*: Let (X, d) be a symmetric space, let $T : X \to CB(X), \phi \in \Phi$, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ .Then T is generalized (α,ϕ,ψ) contractive multifunction if α(x,y)≥1

$$\Rightarrow \phi(H(Tx,Ty)) \le \psi\left(\max\begin{Bmatrix}\phi\circ d(x,y), \phi\circ d(x,Tx), \phi\circ d(y,Ty), \\ \frac{1}{2}[\phi\circ d(x,Ty)+\phi\circ d(y,Tx)]\end{Bmatrix}\right) \text{ for } x, y\in X.$$

Now we state and prove our main results. In the following result, we use the argument similar to that in the corollary 3.5[2].The main results consist of the existence and uniqueness of fixed point for generalized (α,ϕ,ψ) contraction multivalued maps in semi-metric spaces.

III. Main results

Theorem A:

Let (X, d) be d-bounded and S-complete semi-metric space, $\alpha: X\times X\to[0,\infty)$ be a function ,$\phi\in\Phi$ and $\psi\in\Psi$ and $T: X\to CB(X)$ be a closed-valued multifunction,triangular α-admissible and (α,ϕ,ψ) contractive multifunction on X such that $\phi(H(Tx,Ty)) \le \psi(\phi(d(x,y)))$ for all $x,y\in X$. Suppose that there exists $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\ge 1$ Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all n and $x_n\to x$, then, $\alpha(x_n,x)\ge 1$ for all n. Then T has a fixed point.

Proof:

Let $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\ge 1 \Rightarrow \alpha(Tx_0,Tx_1)\ge 1$. Define the sequence $\{x_n\}$ in x by $x_{n+1}=Tx_n$ for all n≥0. So $\alpha(x_n,x_{n+1})\ge 1$. Since T is triangular α-admissible then $\alpha(x_n,x_{n+1})\ge 1$ and $\alpha(x_{n+1},x_{n+2})\ge 1 \Rightarrow \alpha(x_n,x_{n+2})\ge 1 \Rightarrow \alpha(Tx_n,Tx_{n+2})\ge 1$.

Then by induction, we get $\alpha(x_n,x_m)\ge 1$ for all m>n≥0.So $\alpha(Tx_n,Tx_m)\ge 1$. For all n=0, 1, 2… we denote $\delta_n = \sup_{j,k\in N}\phi(d(x_{n+j},x_{n+k}))$

Since X is d-bounded and the fact that ϕ is non-decreasing function, we have $\delta_n<\infty$, for all n=0,1,2,…By continuing this process, $d(x_{n+j},Tx_{n+j-1})=0$, $d(x_{n+k},Tx_{n+k-1})=0$ for all n,j,k∈N.It follows ,

$$d(x_{n+j},x_{n+k}) = d(Tx_{n+j-1},Tx_{n+k-1}) \le \phi(H(Tx_{n+j-1},Tx_{n+k-1})) \le \psi(\phi\circ d(x_{n+j-1},x_{n+k-1})) = \psi(\delta_{n-1})$$

$d(x_{n+j},x_{n+k}) \le \phi\circ H(Tx_{n+j-1},Tx_{n+k-1}) \le \psi(\delta_{n-1})$ for all j<k.

Since ψ is non-decreasing function then $d(x_{n+j},x_{n+k}) \le \phi\circ H(Tx_{n+j-1},Tx_{n+k-1}) \le \psi(\delta_{n-1})$ for all j<k.

By symmetry of d, we get $d(x_{n+j},x_{n+k}) \le \phi\circ H(Tx_{n+j-1},Tx_{n+k-1}) \le \psi(\delta_{n-1})$ for all j>k.

Also for j = k, we have $d(x_{n+j},x_{n+k}) \le \phi\circ H(Tx_{n+j-1},Tx_{n+k-1}) = \phi(0) = 0 \le \psi(\delta_{n-1})$

Thus $d(x_{n+j},x_{n+k}) \le \phi\circ H(Tx_{n+j-1},Tx_{n+k-1}) \le \psi(\delta_{n-1})$ for all j,k∈N. So we have $\delta_n = \psi^n(\delta_0)$ for all n∈N . Now we have $d(x_n,x_m) \le \phi(H(Tx_{n-1},Tx_{m-1}) \le \delta_{n-1} \le \psi^{n-1}(\delta_0)$ for all n,m≥1.

This implies that $\lim_{n\to\infty} d(x_n,x_m)=0$ which shows that $\{x_n\}$ is a d- Cauchy sequence in X Since X is S-complete, $x_n\to x^*$ for some $x^*\in X$. Since $\alpha(x_n,x^*)\ge 1$ for all n which implies that $\alpha(Tx_n,Tx^*)\ge 1$ for all n, thus $d(x_{n+1},Tx^*) = d(Tx_n,Tx^*) \le \phi(H(Tx_n,Tx^*)) \le \psi(\phi(d(x_n,x^*)))$

Letting n→∞, we get $d(x^*,Tx^*)=0$. So T has a fixed point.

Theorem B:

Let (X, d) be d-bounded and S-complete symmetric space, $\alpha: X\times X\to[0,\infty)$ be a function ,$\phi\in\Phi$ and $\psi\in\Psi$ and $T: X\to CB(X)$ be a closed-valued multifunction,triangular α-admissible and (α,ϕ,ψ) contractive multifunction on X such that

$\phi(H(Tx,Ty)) \le \psi(\phi(d(x,y)))$ for all $x, y \in X$. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$. If T is τ_d - continuous and (X, d) satisfies the property (WC), then T has a fixed point.

Proof:

Following the proof of theorem A, there exists a d- Cauchy sequence $\{x_n\}$ in X and since X is S-complete, there exists $x^* \in X$ as n→∞ in the topology τ_d.

Since T is τ_d-continuous, then $Tx_n = Tx^*$ in τ_d and so $\lim_{n\to\infty} d(Tx_n, Tx^*) = 0$. Since (X, d) satisfies the property (WC), we have $d(x^*,Tx^*) \le \liminf_{n\to\infty} d(x_{n+1},Tx_n) = 0$ which implies that $d(x^*,Tx^*) = 0$. ∴T has a fixed point in X.

Theorem C:

Let (X, d) be d-bounded and S-complete symmetric space satisfying (W.4), $\alpha : X\times X \to [0,\infty)$ be a function, $\phi \in \Phi$ and $\psi \in \Psi$ and $T: X \to CB(X)$ be a closed-valued multifunction, triangular α-admissible and (α,ϕ, ψ) contractive multifunction on X such that $\phi(H(Tx,Ty)) \le \psi(\phi(d(x,y)))$ for all $x, y \in X$. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$. Then T has a fixed point.

Proof:

Following the proof of theorem A, there exists a d- Cauchy sequence $\{x_n\}$ in X and since X is S-complete, there exists $x^* \in X$ as n→∞ in the topology τ_d.

Let $\varepsilon > 1$.From Lemma 6 , for each n∈{1,2,...} there exists $y_n \in Tx^*$ such that $d(x_{n+1}, y_n) \le \varepsilon H(Tx_n, Tx^*) \le \varepsilon\psi(\phi(d(x_n, x^*))), n = 1,2,...$ Which implies that $\lim_{n\to\infty} d(x_{n+1}, y_n)$ =0. In view of (W.4), we have $\lim_{n\to\infty} d(y_n, x^*)$ =0 and therefore $x^* \in \overline{Tx^*}^d$ =Tx*. So x*=Tx* . Therefore T has a fixed point.

Example:

Let $X = [0,\infty)$, $d(x,y) = (x-y)^2$ and $\delta \in (0,1)$ be a fixed numbers. Define $T: X \to 2^X$ by $Tx = [0, \delta x]$ for all x∈X and $\alpha : X\times X \to [0,\infty)$ by α(x, y)=1 whenever $x, y \in [0,1]$ and α(x, y)=0 whenever x ∉[0, 1] or y ∉ [0, 1]. Now, we show that T is α- admissible. If α(x, y)≥1, then x, y∈[0, 1] and so Tx and Ty are subsets of [0, 1]. Thus a, b∈[0, 1] for all a ∈Tx and b ∈Ty. This implies that $\alpha(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\} = 1$. Therefore, T is α - admissible. Now we show that T is an (α,ϕ, ψ) contractive multifunction, where ψ(t)=δt for all t ≥0 and $\phi(t) = \sqrt{t}$ for all t ≥0. If $x \notin \left[0, \frac{1}{\delta}\right]$ or $y \notin \left[0, \frac{1}{\delta}\right]$, then it is easy to show that $\alpha(Tx,Ty) = 0$. If $0 \le x, y \le \frac{1}{\delta,}$ then $\alpha(Tx,Ty) = 1$By using the definition of the Hausdorff metric, it is easy to see that $H(Tx,Ty) \le \delta d(x,y)$ for $x, y \in \left[0, \frac{1}{\delta}\right]$. Thus, $\phi(H(Tx,Ty)) \le \psi(\phi(d(x,y)))$ for all x, y∈X. Therefore, T is an (α,ϕ,ψ) contractive multifunction.

Theorem D:

Let (X, d) be a d-bounded and S-complete symmetric space, $\alpha : X\times X \to [0,\infty)$ be a function, $\phi \in \Phi$ and $\psi \in \Psi$, $T: X \to CB(X)$, be a closed-valued multifunction, triangular α-admissible and (α,ϕ, ψ) contractive multifunction on X such that

$$\phi(H(Tx,Ty)) \le \psi\left(\max\left\{\begin{matrix}\phi\circ d(x,y),\phi\circ d(x,Tx),\phi\circ d(y,Ty),\\ \frac{1}{2}[\phi\circ d(x,Ty)+\phi\circ d(y,Tx)]\end{matrix}\right\}\right)$$ for all x,y∈X satisfying α(x,y)≥1. Suppose the following conditions hold:

1. T is triangular α-admissible.
2. There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0)\ge 1$.
3. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all n≥0 and $\lim_{n\to\infty} d(x_n,x)=0$, then $\alpha(x_n,x)\ge 1$ for all n≥0.

Then T has fixed point in X..

Proof:

Let $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1)\ge 1 \Rightarrow \alpha(Tx_0,Tx_1)\ge 1$. Define the sequence $\{x_n\}$ in x by $x_{n+1}=Tx_n$ for all n≥0. So $\alpha(x_n,x_{n+1})\ge 1$. Since T is triangular α-admissible then $\alpha(x_n,x_{n+1})\ge 1$ and $\alpha(x_{n+1},x_{n+2})\ge 1 \Rightarrow \alpha(x_n,x_{n+2})\ge 1 \Rightarrow \alpha(Tx_n,Tx_{n+2})\ge 1$.

Then by induction, we get $\alpha(x_n,x_m)\ge 1$ for all m>n≥0.So $\alpha(Tx_n,Tx_m)\ge 1$. For all n=0, 1, 2… we denote $\delta_n = \sup_{j,k\in N}\phi\left(d\left(x_{n+j},x_{n+k}\right)\right)$. Since X is d-bounded and the fact that ϕ is non-decreasing function, we have $\delta_n<\infty$, for all n=0,1,2,…

By continuing this process, $d\left(x_{n+j},Tx_{n+j-1}\right)=0$, $d\left(x_{n+k},Tx_{n+k-1}\right)=0$ for all n,j,k∈N.It follows,

$$d\left(x_{n+j},x_{n+k}\right)=d\left(Tx_{n+j-1},Tx_{n+k-1}\right)\le\phi\left(H\left(Tx_{n+j-1},Tx_{n+k-1}\right)\right)\le\psi\left(\max\left\{\begin{matrix}\phi\circ d\left(x_{n+j-1},x_{n+k-1}\right),\phi\circ d\left(x_{n+j-1},Tx_{n+j-1}\right),\phi\circ d\left(x_{n+k-1},x_{n+k-1}\right),\\ \frac{1}{2}\left[\phi\circ d\left(x_{n+j-1},Tx_{n+k-1}\right)+\phi\circ d\left(x_{n+k-1},Tx_{n+j-1}\right)\right]\end{matrix}\right\}\right)$$

$$=\psi\left(\max\left\{\begin{matrix}\phi\circ d\left(x_{n+j-1},x_{n+k-1}\right),\phi\circ d\left(x_{n+j-1},x_{n+j}\right),\phi\circ d\left(x_{n+k-1},x_{n+k}\right),\\ \frac{1}{2}\left[\phi\circ d\left(x_{n+j-1},x_{n+k}\right)+\phi\circ d\left(x_{n+k-1},x_{n+j}\right)\right]\end{matrix}\right\}\right)$$

$d\left(x_{n+j},x_{n+k}\right)\le\phi\circ H\left(Tx_{n+j-1},Tx_{n+k-1}\right)\le\psi(\delta_{n-1})$ for all j<k. Since ψ is non-decreasing function then $d\left(x_{n+j},x_{n+k}\right)\le\phi\circ H\left(Tx_{n+j-1},Tx_{n+k-1}\right)\le\psi(\delta_{n-1})$ for all j<k. By symmetry of d, we get $d\left(x_{n+j},x_{n+k}\right)\le\phi\circ H\left(Tx_{n+j-1},Tx_{n+k-1}\right)\le\psi(\delta_{n-1})$ for all j>k. Also for j = k, we have $d\left(x_{n+j},x_{n+k}\right)\le\phi\circ H\left(Tx_{n+j-1},Tx_{n+k-1}\right)=\phi(0)=0\le\psi(\delta_{n-1})$

Thus $d\left(x_{n+j},x_{n+k}\right)\le\phi\circ H\left(Tx_{n+j-1},Tx_{n+k-1}\right)\le\psi(\delta_{n-1})$ for all j,k∈N. So we have $\delta_n=\psi^n(\delta_0)$ for all n∈N .Now we have $d(x_n,x_m)\le\phi(H(Tx_{n-1},Tx_{m-1})\le\delta_{n-1}\le\psi^{n-1}(\delta_0)$ for all n,m≥1. This implies that $\lim_{n\to\infty}d(x_n,x_m)=0$ which shows that $\{x_n\}$ is a d- Cauchy sequence in X.Since X is S-complete, $x_n\to x^*$ for some $x^*\in X$. Since $\alpha\left(x_n,x^*\right)\ge 1$ for all n which implies that $\alpha\left(Tx_n,Tx^*\right)\ge 1$ for all n, thus $d\left(x_{n+1},Tx^*\right)=d\left(Tx_n,Tx^*\right)\le\phi\left(H\left(Tx_n,Tx^*\right)\right)\le\psi\left(\phi\left(d\left(x_n,x^*\right)\right)\right)$

Letting n→∞, we get $d\left(x^*,Tx^*\right)=0$. So T has a fixed point.

Theorem E:

Let (X,d) be a d-bounded and S-complete symmetric space and $\phi\in\Phi$, $\psi\in\Psi$ and $\alpha: X\times X\to[0,\infty)$, $T:X\to CB(X)$ such that

$$\phi\circ H(Tx,Ty)\le\psi\left(\max\left\{\begin{matrix}\phi\circ d(x,y),\phi\circ d(x,Tx),\phi\circ d(y,Ty),\\ \frac{1}{2}[\phi\circ d(x,Ty)+\phi\circ d(y,Tx)]\end{matrix}\right\}\right)$$ for all x,y∈X satisfying α(x,y)≥1. Suppose the following conditions hold:

1. T is triangular α-admissible.
2. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.
3. T is τ_d continuous.
4. (X, d) satisfies the property (WC).

Then T has fixed point in X.

Proof:

Following the proof of theorem C, there exists a d- Cauchy sequence $\{x_n\}$ in X .Since X is S-complete, there exists $x^* \in X$ as n→∞ in the topology τ_d .Since T is τ_d -continuous, then $Tx_n = Tx^*$ in τ_d and so $\lim_{n\to\infty} d(Tx_n, Tx^*) = 0$. Since (X, d) satisfies the property (WC), we have $d(x^*, Tx^*) \leq \liminf_{n\to\infty} d(x_{n+1}, Tx_n) = 0$ which implies that $d(x^*, Tx^*) = 0$. ∴T has a fixed point in X.

Now we prove the uniqueness of the fixed pont in the above theorem. For this, we denote the set of fixed points of T by Fix(T).

Theorem F:

Assume that all the hypothesis of Theorem A,B,C,D and E hold. Also suppose $\forall x, y \in Fix(T)$, there exists z∈X such that $\alpha(x,z) \geq 1, \alpha(y,z) \geq 1$ holds, then the fixed point of T is unique.

Proof:

Suppose there exists u, w∈X such that $d(u., Tu) = d(w, Tw) = 0$. Now by the assumption, we have $\alpha(u,w) \geq 1$, it follows $\alpha(Tu, Tw) \geq 1$.Then

$$\phi(H(Tu,Tw)) = \phi(H(u,w)) \leq \psi\left(\max\left\{\begin{array}{l}\phi\circ d(u,w), \phi\circ d(u,u), \phi\circ d(w,w),\\ \frac{1}{2}[\phi\circ d(u,w) + \phi\circ d(w,u)]\end{array}\right\}\right)$$

$$= \psi(\max\{\phi\circ d(u,w), \phi(0)\}) = \psi(\phi\circ d(u,w))$$

Which implies that $\phi\circ d(u,w) = 0$ and so u=w.

Corollary G:

Let (X,d) be a d-bounded and S-complete symmetric space satisfying (W4) and $T: X \to CB(X)$, $\phi \in \Phi$, $\psi \in \Psi$ and $\alpha: X\times X \to [0,\infty)$ such that $\phi(H(Tx,Ty)) \leq \psi\left(\max\left\{\begin{array}{l}\phi\circ d(x,y), \phi\circ d(x,Tx), \phi\circ d(y,Ty),\\ \frac{1}{2}[\phi\circ d(x,Ty) + \phi\circ d(y,Tx)]\end{array}\right\}\right)$ for all x,y∈X.Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$ Then T has fixed point in X..

Corollary H:

Let (X,d) be a d-bounded and S-complete symmetric space and $\alpha: X\times X \to [0,\infty)$, $T: X \to CB(X)$, $\psi \in \Psi$ such that $H(Tx,Ty) \leq \psi\left(\max\left\{\begin{array}{l}d(x,y), d(x,Tx), d(y,Ty),\\ \frac{1}{2}[d(x,Ty) + d(y,Tx)]\end{array}\right\}\right)$ for all x,y∈X. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is τ_d continuous then T has fixed point of X.

Corollary I:

Let (X,d) be a d-bounded and S-complete symmetric space satisfying (W4) and $\phi \in \Phi$, $\psi \in \Psi$ and $T : X \to CB(X)$, such that $\phi(H(Tx,Ty)) \le \psi\left(\max\left\{\begin{matrix}\phi(d(x,y)),\phi(d(x,Tx)),\phi(d(y,Ty)),\\ \frac{1}{2}\left[\phi(d(x,Ty))+\phi(d(y,Tx))\right]\end{matrix}\right\}\right)$ for all x,y∈X.

Then T has fixed point in X.

Proof: It suffices to take $\alpha(x,y)=1$ in Theorem D. The uniqueness of z holds since the condition in Theorem F is satisfied.

Corollary J:

Let (X,d) be a d-bounded and S-complete symmetric space satisfying (W4) and $T : X \to X$, $\phi \in \Phi$, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ such that $\phi(d(Tx,Ty)) \le \psi\left(\max\left\{\begin{matrix}\phi\circ d(x,y),\phi\circ d(x,Tx),\phi\circ d(y,Ty),\\ \frac{1}{2}\left[\phi\circ d(x,Ty)+\phi\circ d(y,Tx)\right]\end{matrix}\right\}\right)$ for all x,y∈X. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$ and T is triangular α-admissible . Then T has fixed point in X.

Corollary K :

Let (X,d) be a d-bounded and S-complete symmetric space and $T : X \to X$, $\psi \in \Psi$ such that $d(Tx,Ty) \le \psi\left(\max\left\{\begin{matrix}d(x,y),d(x,Tx),d(y,Ty),\\ \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\end{matrix}\right\}\right)$ for all x,y∈X. If T is τ_d continuous then T has fixed point of X.

IV. Conclusion:

Recently many results appeared in the literature giving the problems related to the fixed point for multivalued maps.In this paper we obtained the results for existence of the fixed points of multivalued maps that satisfying a generalized contractive condition.As a consequence we obtained some fixed point for multivalued contraction. We present an example and some corollaries.

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