

# RANGE SPACE AND NULL SPACE OF (3, 2) JECTION OPERATOR

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**ABSTRACT** : In the present paper, our objective is to study range space and null space of a special type of operator, called a (3, 2)-jection operator which is a suitable generalization of projection.

**Key words** : Range of an operator, Null space of an operator, Projection, (3, 2)-jection operator..

## I. INTRODUCTION & PRILIMINARIES :

Let  $L(V, W)$  denote the space of all linear mapping from a vector space  $V$  into another vector space  $W$ . Let  $T \in L(V, W)$  where  $V$  and  $W$  are finite dimensional vector space. Then the set  $R_T$  of vectors  $w \in W$  such that  $Tv = w$  for some  $v \in V$ , is called the range space of  $T$ , and the set  $N_T$  of vectors  $v \in V$  for which  $Tv = 0$  is called the null space of  $T$ . Sometime, we refer the space  $N_T$  and  $R_T$  as the kernel of  $T$  and the image of  $T$  respectively.

Our work in the present paper centres around a special type of operator, called a (3, 2)-jection operator.

In linear algebra and functional analysis a projection is of fundamental importance, which is defined as a linear operator  $E$  on a vector space  $X$  such that  $E^2 = E$ . That is, wherever  $E$  is applied twice to any element  $x \in X$ , it gives the same result as if it were applied once.

As stated above, projection is a special case of idempotent. On the basis of above definition of projection, we develop a new operator called a (3, 2)-jection operator, which is a suitable generalization of projection, defined as a linear operator on a linear space  $X$  such that  $E^3 = E^2$ . This definition of (3, 2)-jection operator can be carried over verbaton to Hilbert space  $H$  with an additional condition that  $E^* = E$  where  $E^*$  stands for adjoint of  $E$ .

## II. MAIN RESULT :

The analogous results are listed with the following theorems :

### Theorem 2.1 :

Let  $E_1$  and  $E_2$  be two (3, 2)-jections on a linear space  $L$  then

$$R_{E_2^2} \subseteq R_{E_1^2} \Leftrightarrow E_1^2 E_2^2 = E_2^2$$

**Proof** : we suppose first  $R_{E_2^2} \subseteq R_{E_1^2}$  and prove  $E_1^2 E_2^2 = E_2^2$

$$\text{Let, } R_{E_2^2} \subseteq R_{E_1^2}$$

$$\text{If, } z \in L, E_2^2 z \in R_{E_2^2} \subseteq R_{E_1^2}$$

$$\Rightarrow E_2^2 z \in R_{E_1^2}$$

$$\text{so, } E_2^2 z = E_1^2 y \text{ for some } y \in L$$

$$\Rightarrow E_1^2 (E_2^2 z) = E_1^2 (E_1^2 y) = E_1^4 y = E_1^2 y = E_2^2 z$$

$$\text{i.e. } E_1^2 E_2^2 z = E_2^2 z \text{ for any } z \in L$$

$$\text{Hence, } E_1^2 E_2^2 = E_2^2$$

Conversely,

Here we suppose  $E_1^2 E_2^2 = E_2^2$  and prove  $R_{E_2^2} \subseteq R_{E_1^2}$

$$\text{Let, } E_1^2 E_2^2 = E_2^2$$

$$\text{Let } z \in R_{E_2^2} \Rightarrow z = E_2^2 z_1 \text{ for some } z_1 \text{ in } L$$

$$\Rightarrow E_1^2 z = E_1^2 (E_2^2 z_1) = E_1^2 E_2^2 z_1 = E_2^2 z_1 = z$$

$$\Rightarrow z = E_1^2 z$$

$$\Rightarrow z \in R_{E_1^2}$$

$$\text{Hence } R_{E_2^2} \subseteq R_{E_1^2}$$

**Theorem 2.2 :**

If  $E_1, E_2$  are two commuting (3, 2)-jections on a linear space  $L$ ,

$$\text{then } N_{E_1}^2 \subseteq N_{E_2}^2 \Leftrightarrow E_1^2 E_2^2 = E_2^2$$

**Proof :** Let  $N_{E_1}^2 \subseteq N_{E_2}^2$  Let  $z \in L$

$$\begin{aligned} \text{Now, } E_1^2(z - E_1^2 z) &= E_1^2 z - E_1^2(E_1^2 z) = E_1^2 z - E_1^4 z \\ &= E_1^2 z - E_1^2 z = 0 \end{aligned}$$

$$\Rightarrow z - E_1^2 z \in N_{E_1}^2 \subseteq N_{E_2}^2$$

$$\therefore z - E_1^2 z \in N_{E_2}^2 \Rightarrow E_2^2(z - E_1^2 z) = 0$$

$$\Rightarrow E_2^2 z - E_2^2(E_1^2 z) = 0$$

$$\Rightarrow E_2^2 z - E_2^2 E_1^2 z = 0$$

$$\Rightarrow E_2^2 z = E_2^2 E_1^2 z$$

$$= E_1^2 E_2^2 z \text{ \{because } E_1 \text{ and } E_2 \text{ are commuting}\}$$

But  $z$  is arbitrary, so

$$E_2^2 = E_1^2 E_2^2$$

Conversely,

Here we suppose  $E_1^2 E_2^2 = E_2^2$  and prove  $N_{E_1}^2 \subseteq N_{E_2}^2$

if  $z \in N_{E_1}^2$  then  $E_1^2 z = 0$

.....(2.2.1)

We have,

$$E_2^2 = E_1^2 E_2^2$$

$$\Rightarrow E_2^2 z = E_1^2 E_2^2 z$$

$$= E_2^2 E_1^2 z$$

$$= E_2^2(E_1^2 z)$$

$$= E_2^2(0)$$

$$= 0$$

$$\Rightarrow z \in N_{E_2}^2$$

Hence  $N_{E_1}^2 \subseteq N_{E_2}^2$

**III. CONCLUSION :**

Combining theorems (2.1) and (2.2) if  $E_1, E_2$  are two commuting (3, 2)-jections on a linear space  $L$  then

$$R_{E_2}^2 \subseteq R_{E_1}^2 \Leftrightarrow E_1^2 E_2^2 = E_2^2 \Leftrightarrow N_{E_1}^2 \subseteq N_{E_2}^2$$

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