COMPLETENESS AND COMPACTNESS OF BITOPOLOGICAL SPACE

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Abstract : Compactness in bitopological space is analogous to classical concept. A bitopological space is called *p*-compact if every *p*-open cover of the space has a finite subcover, while in (1971) Swart define as a cover *c* of the bitopological space (*X*, *u*, *v*) is called *uv*-open if $c \subset u, v$ and in 1972, Datta defined *s*-compact space if every *uv*-open cover of the space has a finite subcover. In this paper we are concerned with the idea of completeness and compactness in bitopological spaces.

Keywords : bitopological space, paracompactness, bifilter, completeness and compactness.

Introduction

In this paper we are concerned with the idea of completeness and compactness in bitopological spaces. Kelly [7] was the first mathematician who studied bitopological spaces. Several others mathematician defined completeness, compactness for such spaces such as Kim [8], Fletcher, Hoyle and Patty [5], Birsan [2], Datta [4], Swart [10] and Pahk and Choi [9]. In the year 2010, Bose and Mukharjee [3] defining pairwise paracompactness and Abushaheen and Hdeib (2016) [1] introduce the concept of [*a*, *b*]-compactness.

Kelly [7], in 1963, defined *p*-regular and *p*-normal spaces. Whereas in 1969, Fletcher et.al. [5] gave the definitions of *uv*-open and *p*-open covers in a bitopological space. We define cover of a bitopoplogical space as follow :

A cover *U* of a bitopological space (*X*, *u*, *v*) is called *uv*-open if $U \subset u \cup v$. If, in addition, *U* contains at least one non-empty member of *u* and at least one non-empty member of *v*, it is called *p*-open. Also they defined the concept of *p*-compact space as follows:

Compactness in bitopological space is analogous to classical concept. A bitopological space is called *p*-compact if every *p*-open cover of the space has a finite subcover, while in (1971) Swart [10] define as a cover *c* of the bitopological space (*X*, *u*, *v*) is called *uv*-open if $c \subset u, v$ and in 1972, Datta [4] defined *s*-compact space if every *uv*-open cover of the space has a finite subcover and in 1969, Birsan [2] gave the following definitions.

A bitopological space (*X*, *u*, *v*) is called *u* compact with respect to *v* if for each *u*-open cover of *X* there is a finite *v*-open subcover.

Fora and Hdeib [5], in 1983 introduced the concepts of *p*-Lindelof, *s*-Lindelof and *B*-Lindelof spaces in analogue manner. They also gave the definitions of certain types of functions as follows: A function $f : (X, u, v) \to (Y, u', v')$ is *p*-continuous (*p*-open, *p*-closed, *p*-homeomorphism, respectively) if both $f : (X, u) \to (Y, u')$ and $f : (X, v) \to (Y, v')$ are continuous (open, closed, homeomorphism, respectively).

Convergence of bifilter :

Definition 3.1.1. Let (X, δ) be a cqu. We shall say that (X, δ) is complete if each *D*-regular δ-Cauchy bifilter on *X* is convergent in $(X, t_u(\delta), t_v(\delta))$.

Proposition 3.1.1. Let (X, δ) be a complete cqu, and $\boldsymbol{\beta} \subseteq \delta_0$ a base of δ. Then every $\boldsymbol{\beta}$-regular δ-Cauchy bifilter on *X* is convergent.

Proof. If β is $\boldsymbol{\beta}$-regular and δ-Cauchy, and we construct β^* as in Proposition 1.3.3, then β^* is (minimal) *D*-regular and δ-Cauchy, and hence convergent. However $\beta^* \subseteq \beta^*$ so β is convergent also.

Let δ be an interior separated cqu on *X*, and let $\Gamma(x)$ be any set of δ_0-regular δ-Cauchy bifilters on *X* which includes the set $\{\beta(x) \mid x \in X\}$ of nhd. bifilters on *X*. We denote by *j* the one to one map $j(x) = \beta(x)$ of *X* in to $\Gamma(x)$. We are going to show how we may give $\Gamma(x)$ an interior confluence quasi uniformity $\tilde{\delta}$ with the property that $(\Gamma(x), \tilde{\delta})$ is a strict extension of (X, δ) for the map *j*. For $A \subseteq X$ define

$$A_u = \{\beta \mid \beta \in \Gamma(x), A \in \beta_u\}, \text{ and}$$

$$A_u^0 = \{\beta \mid \beta \in \Gamma(x) \text{ and } \exists\, (d, c) \in \delta_0 \text{ such that } UdV \text{ and } V \in \beta_v \Rightarrow U \subseteq A\}$$

For $B \subseteq X$ we define B_v and B_v^0 in a similar way. Some important properties of these sets are set out in the next lemma.

Lemma 3.1.1. For each $A, B \subseteq X$:

(a) $A_u^0 \subseteq A_u$ a and $B_u^0 \subseteq B_u$

(b) $j^{-1}(A_u^0) = j^{-1}(A_u) = t_u(\delta) - \text{int}[A]$, and

$j^{-1}(B_v^0) = j^{-1}(B_v) = t_v(\delta) - \text{int}[B]$.

Proof. Straightforward.

If *c* is a confluence relation on *X* we may define the confluence relation $\tilde{c}$ on $\Gamma(x)$ by

$$P\tilde{c}Q \Leftrightarrow P_i Q \text{ or } \exists A \in t_v(\delta), B \in t_v(\delta) \text{ with } AcB, \text{ and } A_u^0 \subseteq P \text{ and } B_v^0 \subseteq Q.$$

If *d* is a *c*-dual cover of *X* we define $\tilde{d}$ on $\Gamma(x)$ by

$$P\tilde{c}Q \Leftrightarrow \exists\, UdV \text{ with } P = U_u^0 \text{ and } Q = V_v^0.$$

If $(d, c) \in \delta$ is open then $\tilde{d}$ is a $\tilde{c}$-dual cover of $\Gamma(x)$. That $\tilde{d} \subseteq \tilde{c}$ is clear. To show the uniform covering of $\tilde{d}$ is $\Gamma(x)$ take $\beta \in \Gamma(x)$ and $(e, b) \in \delta_0$ with $(e,b) \prec (*)(d,c)$. Since β is δ-Cauchy we have ReS with $(R, S) \in \beta$. Also we have UdV with $St_b(e, R) \subseteq U$ and $St_b(e, R) \subseteq V$, and it is easy to verify that $\beta \in U_u^0 \cap V_v^0$; while by definition $U_u^0 \tilde{d} V_v^0$.

We may now give:

Theorem 3.1.1. With the notation as above, $\{(\tilde{d},\tilde{c}) \mid (d,c) \in \delta_0\}$ is an open interior base for a confluence quasi-uniformity $\tilde{\delta}$ on $\Gamma(x)$. $(\Gamma(x), \tilde{\delta})$ is a strict extension of (X, δ) under the map j.

Proof. For (d, c), $(e, b) \in \delta_0$ with $(e,b) \prec (*)(d,c)$ take ReS and UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$. If $\mathbf{\beta} \in St_{\tilde{b}}(e, R_u^0)$ we have $R'eS'$ with $\beta \in R_u^0$ and $R_u^0 \; \tilde{b} \; R_v^0$.

There are two cases:

(i) $R_u^0 \cap S_v'^0 \neq \phi$. Then $\exists \mathbf{b} \in R_u^0 \cap S_v'^0$ by Lemma 2.5.1 (a.), and $CR, S') \in \mathbf{b}$ implies RbS' since $\mathbf{b}$ is δ_0-regular.

(ii) $R_u^0 \cap S_v'^0 \neq \phi$. Then $\exists A \in t_u(\delta), B \in t_v(\delta)$ with AbB, $A_u^0 \subseteq R_u^0$ and $B_v^0 \subseteq S_v'^0$. But then $A = t_u(\delta) - \text{int}[A] = j^{-1}(A_u^0) \subseteq j^{-1}(R_u^0) = t_u(\delta) - \text{int}[R] = R$ by Lemma 3.1.1 (b). In the same way $B \subseteq S'$, and so we again have RbS'.

In either case we therefore have $R' \subseteq St_b(e, R) \subseteq U$, and hence $St_{\tilde{b}}(\tilde{e}, R_u^0) \subseteq U_u^0$. In just the same way $St_{\tilde{b}}(S_v^0, \tilde{e}) \subseteq V_v^0$. Finally we clearly have $\tilde{b} \subseteq \tilde{c}$, and hence $(\tilde{e},\tilde{b}) \prec (*)(\tilde{d},\tilde{c})$. This proves that $\{(\tilde{d},\tilde{c}) \mid (d,c) \in \delta_0\}$ is a). base for a cqu $\tilde{\delta}$ on $\Gamma(x)$.

Now let us verify that for $A, B \subseteq X$ we have:

$[j^{-1}(A)]_u^* = (A_u^0) = t_u(\tilde{\delta}) - \text{int}[A_u]$, and

$[j^{-1}(B)]_v^* = B_v^0 = t_v(\tilde{\delta}) - \text{int}[B_v]$,

where the sets on the left are formed for the subset $j(X)$ of the bitopological space $(\Gamma(x), t_u(\tilde{\delta}), t_v(\tilde{\delta}))$.

First take $\beta \in [j(A)]_u^*$, , then $\beta \in G \in t_u(\tilde{\delta})$ with $G \cap j(X) \subseteq j(A)$.

Take $(d, c) \in \delta_0$ with $St(\tilde{d}), \{\beta\}) \subseteq G$, and $(e, b) \in \delta_0$ with $(e,b) \prec (*)(d,c)$. . Since β is δ-Cauchy we have ReS with $(R, S) \in \beta$, and we have UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq$ V. Then $\beta \in V_v^0$ and so $U_u^0 \subseteq St(\tilde{d}), \{\beta\}) \subseteq G$. Hence $U \subseteq A$ since j is one to one. To show that $\beta \in A_u^0$ take $R'eS'$ with $S' \in \beta_v$.

Then $(R, S') \in \beta_v$ so RbS' since β is δ_0-regular. Hence $R' \subseteq St_b(e, R) \subseteq U \subseteq A$, and $\beta \in A_u^0$ as required. This verifies $[j^{-1}(A)]_u^* = (A_u^0)$.

If now $\beta \in A_u^0$ and $(d, c) \in \delta_0$ satisfies UdV, $V \in \beta_v \Rightarrow U \subseteq A$, , then it is easy to verify $St(\tilde{d}), \{\beta\}) \subseteq A_u$, and so $t_u(\tilde{\delta}) - \text{int}[A_u]$, . Hence $A_u^0 \subseteq t_u(\tilde{\delta}) - \text{int}[A_u]$.

Finally it is trivial to verify $t_u(\tilde{\delta}) - \text{int}[A_u] \subseteq [j^{-1}(A)]_u^*$ and so the first set of equalities is proved. The proof of the other equalities is similar.

It follows at once that $\{(\tilde{d},\tilde{c}) \mid (d,c) \in \delta_0\}$ is an open interior base for $\tilde{\delta}$.

It is trivial to verify that $j(X)$ is bidense in $\Gamma(X)$. Also, using the above equalities together with Lemma 3.1.1 we may easily show that the conditions of Definition 2.3.2 (b) are satisfied for the base $\{(\tilde{d},\tilde{c}) \mid (d,c) \in \delta_0\}$ of $\tilde{\delta}$, and so $j(X)$ is strictly $\tilde{\delta}$-embedded in $\Gamma(X)$.

It remains to show that j is an isomorphism of (X, δ) and $(j(X), \tilde{\delta}_{j(X)})$. However this follows at once from the relations

$(d,c) \prec (j^{-1}(\tilde{d}_{j(X)}), (j^{-1}(\tilde{c}_{j(X)})$, and

$(\tilde{d}_{j(X)}, \tilde{c}_{j(X)}) \prec ((j^{-1})^{-1}(d), (j^{-1})^{-1}(c))$

which are easily verified for any $(d, c) \in \delta_0$. This completes the proof of the theorem.

The strict extension $(\Gamma(X), \tilde{\delta})$ constructed above will not in general be separated. In Proposition 2.2.2 (Chapter 2), we may obtain a separated strict extension by taking the associated separated cqu. A second way in which we may obtain a separated extension is to require that $\Gamma(X)$ contain only minimal δ-Cauchy bifilters. We know in any event by the corollary 2 to Proposition 1.3.2 (Chapter 1) that the elements of $j(X)$ do have this property. We use this second method in the following theorem.

Complete separated strict extension :

Theorem 3.1.2. Let $\Gamma(X)$ denote the set of all D-regular minimal δ-Cauchy bifilters on the separated cqu (X, δ). Then $(\Gamma(X), \tilde{\delta})$ is a complete separated strict extension of (X, δ).

Proof. By Theorem 2.5.1 we know that $(\Gamma(X), \tilde{\delta})$ is a strict extension of (X, δ), so it remains only to show it is complete and separated. Let B be a $\tilde{D}$-regu1ar $\tilde{\delta}$- Cauchy bifilter on $\Gamma_0(X)$, and set

$\beta = \{(P,Q) \mid P, Q \subseteq X, (U_u^0, Q_v^0) \in B\}$. .

It is easy to see that β is a D-regular δ-Cauchy bifilter on X. Construct β^* as in Proposition 1.3.3 (Chapter 1) (for the base $\delta_{0'}$ say); then $\beta^* \in \Gamma_0(X)$. Let us show B converges to β^* in $\Gamma_0(X)$. Take $(d, c) \in \delta_0$. Since $\tilde{\delta}$ is a $\tilde{c}$-dual cover of $\Gamma_0(X)$ we have UdV with $\beta^* \subseteq U_u^0 \cap V_v^0$. Hence $U_u^0 \subseteq St(\tilde{d},\{\beta^*\})$ and $V_v^0 \subseteq St(\{\beta^*\},\tilde{d})$. On the other hand $\beta^* \in U_u \cap V_v$ implies $(U,V) \in \beta^* \subseteq \beta$ and so and $(U_u^0, V_v^0) \in B$. Hence $(St(\tilde{d}),\{\beta^*\}), St(\{\beta^*\},\tilde{d}) \in B$ for all $(d, c) \in \delta_0$, and $B \to \beta^*$ as required. This proves that $(\Gamma_0(X),\tilde{\delta})$ is complete.

To show $(\Gamma_0(X),\tilde{\delta})$ is separated take $\beta, \mathbf{b} \in \Gamma_0(X)$ with $\beta \neq \mathbf{b}$. Say, for example, that $\beta_u \not\subseteq \mathbf{b}_u$. Since β is minimal δ-Cauchy and $\beta^* \subseteq \beta$ we have $\beta = \beta^*$. Hence $\exists(d, c) \in \delta_0$ and $U \in$ dom d with $U \in \beta_u$ and $St_c(d, U) \notin \mathbf{b}_u$. Since β is D-regular we may deduce from this that $\mathbf{b} \notin St(\tilde{d},\{\beta\})$. The other cases may be dealt with in the same way, and we deduce that $(\Gamma_0(X), t_u(\tilde{\delta}), t_v(\tilde{\delta}))$ is weakly pairwise T_0. Hence $(\Gamma_0(X),\tilde{\delta})$ is separated, and the proof is complete.

We now give a theorem on the extension of cquc functions. This is a basic step in proving that separated strict completions of separated cqu are unique up to isomorphism.

Uniqueness of extension :

Theorem 2.5.3. Let (X, μ) be a cqu, (Y, δ) a complete separated cqu and A a bidense and strictly μ-embedded subset of X. If the function $f : A \to Y$ is $(\mu_A - \delta)$ cquc, and $f(A)$ is δ-embedded in Y, then f has a unique $(\mu - \delta)$ cquc extension $\bar{f} : X \to Y$.

Proof. Take $x \in X$. Since A is bidense in X, the nhd bifi1ter trace $\beta^A(x)$ is a bifi1ter on A. Hence

$\mathbf{b}(x) = \{(p, Q) \mid (f^{-1}(P), f^{1}(Q)) \in \beta^A(x)\}$

is a bifi1ter on Y.

Let $\boldsymbol{\beta} = \delta_0$ be a base of δ with the properties of Definition 1.3.2 (a) (Chapter 1) for the δ-embedded subset $f(A)$ or Y. We will show that $\mathbf{b}(x)$ is $\boldsymbol{\beta}$-regular and δ-Cauchy. Take any $(d, c) \in \boldsymbol{\beta}$. Then $(f^{-1}(d), f^{1}(c)) \in \mu_A$. Hence if $U \in$ dom d, $V \in$ ran d and $(U, V) \in \mathbf{b}(x)$ then $(f^{-1}(U), f^{1}(V)) \in \beta^A(x)\}$, and so $(f^{-1}(U), f^{1}(c), f^{1}(V)$ since $\beta^A(x)$ is D_A-regular by Proposition 1.3.2 (Chapter 1)**.** Hence UcV by Lemma 1.4.1 (Chapter 1), and so $\mathbf{b}(x)$ is $\boldsymbol{\beta}$-regular. Next take $(e, b) \in \mu_0$ with $(e_A, b_A) \prec (f^{-1}(d), f^{-1}(c))$, and ReS with $X \in R \cap S$. Since A is bidense $(R \cap A)e_A(S \cap A)$ so we have $U'dV'$ with $(R \cap A) \subseteq f^{-1}(U')$ and $(S \cap A) \subseteq f^{-1}(V')$. However we also have $(R \cap A, S \cap A) \in \beta^A(X)$ and so $(U', V') \in \mathbf{b}(x)$. Thus $\mathbf{b}(x)$ is δ-Cauchy.

It follows by Proposition 1.3.1 (Chapter 1) that $\mathbf{b}(x)$ converges in Y, and the limit is unique since Y is separated. We denote this limit by $\bar{f}(x)$, and in this way we have de fined a function $\bar{f} : X \to X$.

It is clear that if $x \in A$ then $\mathbf{b}(x) \to f(x)$ so in this case $\bar{f}(x) = f(x)$. Hence $\bar{f}$ is an extension of f.

We now show that $\bar{f} : X \to X$ is $(\mu - \delta)$ cquc. Take $(d, c) \in \delta_0$ and $(e, b) \in \boldsymbol{\beta}$ with $(e,b) \prec (*)(d,c)$.. Since A is strictly μ-embedded in X we also have a base $y \subseteq \mu_0$ with the properties of Definition 2.3.2 (b), and since $(f^{1}(e), f^{1}(b)) \in \mu_A$ we have $(g, a) \in y$ with $(g_A, a_A) \prec (f^{1}(e), f^{1}(b))$. Let us verify $(g,a) \prec \bar{f}^{-1}(d), \bar{f}^{-1}(c)$, from which the required result follows.

First take LgK. Since g is open we have $(L \cap A)g_A(K \cap A)$ so we have ReS with $L \cap A \subseteq f^{-1}(R)$ and $K \cap A \subseteq f^{-1}(S)$. Take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \in V$. If $x \in L$ then $L \cap A \subseteq \beta_u^A(x)$ and so $f(L \cap A) \in \mathbf{b}_u(x)$. But $f(L \cap A) \subseteq R$, so we also have $R \in \mathbf{b}_u(x)$. On the other hand there exists, $R'eS'$ with $\bar{f}(x) \in R' \cap S'$.

In particular $S' \in \mathbf{b}_v(x)$. since $\mathbf{b}(x) \to f(x)$. Hence $(R, S') \in \mathbf{b}(x)$, and so RbS' since $\mathbf{b}(x)$ is $\boldsymbol{\beta}$-regular. Hence $\bar{f}(x) \in R' \subseteq St_b(e,R) \subseteq U$, and we have shown $L \subseteq \bar{f}^{-1}(U)$. In just the same way we have $K \subseteq \bar{f}^{-1}(V)$, so it remains only to show $\subseteq \bar{f}^{-1}(c)$.

By Lemma 1.3.1 (Chapter 1)it will suffice for us to verify that $a_A \subseteq [\bar{f}^{-1}(c)]_A$. To this end take Pa_AQ with $P \cap Q = \phi$. Then $P\bar{f}^{-1}(b)Q$ as $a_A \subseteq \bar{f}^{-1}(b)$ so $\exists P' \in t_u(\delta), Q' \in t_v(\delta)$, with $P'bQ', \phi \neq f^{-1}{}_{(P')} \subseteq P$ and $\phi \neq f^{-1}(Q') \subseteq Q$. Since $\bar{f}$ is the same as f on A we deduce that $\bar{f}^{-1}(P') \cap A \subseteq P$ and $\bar{f}^{-1}(Q') \cap A \subseteq Q$. On the other hand $P'cQ'$ as $b \subseteq c$, and so $\bar{f}^{-1}(P')\bar{f}^{-1}(c)\bar{f}^{-1}(Q')$. Hence if we can show $\bar{f}^{-1}(P') \in t_u(\mu)$ and $\bar{f}^{-1}(Q') \in t_v(\mu)$ we shall have

$(\bar{f}^{-1}(P') \cap A)(\bar{f}^{-1}(c))_A(\bar{f}^{-1}(Q') \cap A),$

that is $P(\bar{f}^{-1}(c))_A Q$ as required. To show $\bar{f}^{-1}(P') \in t_u(\mu)$ take $z \in \bar{f}^{-1}(P')$, that is $\bar{f}^{(z) \in P'}$. Now take $(h, q) \in \boldsymbol{\beta}$ with $St(h, \bar{f}(x)) \subseteq P', (k, p) \in \boldsymbol{\beta}$ with $(k, p) \prec (*)(h,q) \in \beta$, and $(m, s) \in \gamma$ with $(m_A, s_A) \prec (f^{-1}(k), f^{-1}(p))$. Then repeating the argument used above we see that $St(m,\{z\}) \subseteq \bar{f}(P')$, and hence $\bar{f}(P') \in t_u(\mu)$. Likewise $\bar{f}(Q') \in t_v(\mu)$ and we have completed the proof that $a \subseteq \bar{f}^{-1}(c)$. Hence $\bar{f}$ is $(\mu - \delta)$ cquc.

Let us prove finally that the extension $\bar{f}$ is unique. Suppose that $\breve{f} : X \to Y$ is also cquc, and that $\breve{f}(x) = \bar{f}(x) = f(x)$ for all $x \in A$. Suppose that for some $x \in X$ we have $\breve{f}(x) \neq \bar{f}(x)$. Since (Y, δ) is separated we see from Proposition 1.2.12 (c) (Chapter 1) that there exists $(d, c) \in \boldsymbol{\beta}$, $U \in$ dom d, $V \in$ ran d with $U \not c V$ and, say, $U \in \beta_u(\bar{f}(x), V \in \beta_v(f(x))$. Since cquc functions are bicontinuous we have $\bar{f}^{-1}(U) \cap A, \bar{f}^{-1}(V) \cap A \in \beta(x)$. It follows at once that $(U, V) \in \mathbf{b}(x)$, and so UcV since $\mathbf{b}(x)$ is β-regular. This contradiction shows $\bar{f} = \breve{f}$, and completes the proof of the theorem.

We may now state our uniqueness theorem.

Theorem 3.1.4. Let (X, δ) be a separated cqu. Let (Y, μ) and (Z, τ) be separated strict completions of (X, δ) with respect to the maps j and k respectively. Then (Y, μ) and (Z, τ) are cqu isomorphic.

Proof. Let $h : j(X) \to Z$ be the map $h = koj^{-1}$, and $t : k(X) \to Y$ the map $t = jok^{-1}$. Since the conditions of Theorem 3.1.3 are satisfied for these maps we have cquc extensions $\bar{h} : Y \to Z$ and $\bar{t} : Z \to Y$. We complete the proof by showing that $\bar{t} = \bar{h}^{-1}$.

Take $y \in Y$, and let $\bar{h}(y) = z \in Z$. Then with the notation as in the proof of Theorem 3.1.3 we have $\mathbf{b}(y) \to z$, and so $\beta(z) \subseteq \mathbf{b}(y)$. If we set $\bar{t}(z) = y'$, then $\mathbf{b}(z) \to y'$ and so $\beta(y') \subseteq \mathbf{b}(z)$. We wish to show that $y = y'$.

Suppose that $y \neq y'$, and suppose $j(X) = t(k(X))$ is (strictly) μ-embedded in Y relative to the base $\boldsymbol{\beta} \subseteq \mu_0$. Since (Y, μ) is separated we have by Proposition 1.2.12 (c) (Chapter 1) that there exists $(d, c) \in \boldsymbol{\beta}$, $U \in$ dom d and $V \in$ ran d with $st_c(d,u) \not\subset St_c(V,d)$ and, say, $U \in \beta_u(y')$ and $V' \in \beta_v(y)$. In particular we have $G \in t_u(\tau)$ with

$$z \in G \text{ and } t(G \cap k(X) \subseteq U \qquad (3.1)$$

Now suppose $k(X) = h(j(X))$ is (strictly) τ-embedded in Z relative to the base $\gamma = \tau_0$. We have $(j^{-1}(d), j^{-1}(c)) \in \delta$ so there exists $(e, a) \in \gamma$ with

$$(k^{-1}(e), k^{-1}(a)) \prec (j^{-1}(d), j^{-1}(c)) \qquad (3.2)$$

and $\quad St(e, \{z\}) \subseteq G \qquad (3.3)$

Finally $(k^{-1}(e), (k^{-1}(a)) \in \delta$ so there exists $(g, b) \in \beta$ with

$$(j^{-1}(g), j^{-1}(b)) \prec (k^{-1}(e), k^{-1}(a)) \qquad (3.4)$$

and $St(\{y\}, g) \subseteq V \qquad (3.5)$

Take ReS and LgK with $z \in R \cap S$ and $y \in L \cap K$. From (3.1) we have

$$t(R \cap k(X)) \subseteq U \qquad (3.6)$$

and from (3.5) we have

$K \subseteq V \quad (3.7)$

Also $K \cap j(X)) \in \underset{v}{\beta}\, j(X)_{(y)}$ so

$$h(K \cap j(X)) \subseteq \mathbf{b}_v(y) \qquad (3.8)$$

Now $j^{-1}(L)\ j^{-1}(g)\ j^{-1}(K)$ so by (3.4) we have $R'eS'$ with

$$j^{-1}(L) \not\subset k^{-1}(R') \text{ and } j^{-1}(K) \subseteq k^{-1}(S') \qquad (3.9)$$

From (3.9) we have $h(K \cap j(X) \subseteq S' \cap k(X)$ and so $S' \in \beta_v(y)$ from (3.8). On the other hand $R \in \beta_v(y)$ and so $(R, S') \in \beta_v(y)$. However $b(y)$ is γ-regular and so

$RaS' \qquad (3.10)$

Now $k^{-1}(R')k^{-1}(e)k^{-1}(S')$ so by (3.2) we have $U'dV'$ with

$$k^{-1}(R') \subseteq j^{-1}(U') \text{ and } k^{-1}(S') \subseteq j^{-1}(V') \qquad (3.11)$$

Since $R \in t_u(\tau)$ and $S' \in t_v(\tau)$ we have $k^{-1}(R)k^{-1}(a)k^{-1}(S')$ by (3.10), so using (3.11), (3.6) and the fact (from (3.2)) that $k^{-1}(a) \subseteq j^{-1}(c)$, we deduce that $j^{-1}(U)\ j^{-1}(c)\ j^{-1}(V')$. However $j(X)$ is, in particular, μ-embedded in Y so from Lemma 1.4.1 (a) (Chapter 1) we see:

$UcV' \qquad (3.10)$

On the other hand from (3.9), (3.11) and (3.7) we have $L \cap j(X) \subseteq U' \cap j(X)$ and $K \cap j(X) \subseteq V \cap j(X)$. We deduce that $j^{-1}(U')\ j^{-1}(b)\ j^{-1}(V)$ and hence from (3.2) and (3.4) we have $j^{-1}(U')j^{-1}(c)j^{-1}(V)$. Using Lemma 1.4.1 (a) (Chapter 1) again gives $U'cV$, and so $V' \subseteq St_c(V, d)$. But also $U \subseteq St_c(d, U)$ and we have the contradiction $St_c(d, U)c\ St(V, d)$. Hence $y = y'$ and $\bar{h}^{-1} = t^{-1}$, which completes the proof.

Corollary. The complete separated strict extension of a separated cqu is unique up to isomorphism.

References

[1] Abushaheen, F.A. and Hdeib, H.Z. (2016) : On [*a*, *b*]-compactness in bitopological spaces, International Journal of Pure and Applied Mathematics, Vol. 110, No. 3, 519-535

[2] Birsan, T. (1969) : Compacite dans les espaces ditopologiques, St. Univ. Iasi, S.I.A., Matematica, 15, 317-328.

[3] Bose, M.K. and Mukharjee, A. (2010) : Pairwise closure-preserving collections and pairwise paracompactness, Mat. Vesnik 62, 299–309.

[4] Datta, M.C. (1972) : Projective bitopological spaces, J. Austral Math. Soc., 13, 327-334.

[5] Fletcher, Hoyle, P.H.B. III and Patty, C.W. (1969) : The comparison of topologies, Duke Math. J. 36, 325–331.

[6] Fora, A. and Hdeib, H. (1983) : On pairwise Lindelof spaces, Rev. Colombiana de Math., 17, 37-58.

[7] Kelly, J.C. (1963) : Bitopological spaces, Proc. Lond. Math. Soc. (3) 13, 71–89.

[8] Kim, Y.W. (1968) : Pairwise compactness. Publ. Math. Debrecen 15:87-90.

[9] Pahk, D.H. and Choi, B.D. (1971) : Notes on pairwise compactness, Kyungpook Math. J., 11, 45-52.

[10] Swart, J. (1971) : Total disconnectedness in bitopological spaces and product bitopological spaces, Indag. Math. 33, 135–145.