LOGICAL MODEL OF QUANTUM MECHANICS

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Abstract : J. von Neumann, the father of quantum logical model of Quantum Mechanics, introduced the idea of a logical implication of physical properties and established that it is the relation between these properties on the one side and the projection operator on a Hilbert space on the other side that should make it possible to obtain a logical model of quantum system. In this paper, our aim is to study to the logic of quantum mechanics.

Keywords : Hilbert space, axiomatisation, orthologic, Quantum mechanics, logical model.

0. Introduction

J. von Neumann is known as the father of quantum logical model of Quantum Mechanics. It is only a very short paragraph in chapter 3 of his book [26] that forms the birth of quantum logic. He introduced the idea of a logical implication of physical properties and established that it is the relation between these properties on the one side and the projection operator on a Hilbert space on the other side that should make it possible to obtain a logical model of quantum system. However, he had to wait till 1936 when J. von Neumann co-authored a paper with G. Birkhoff to establish a full analysis of logical model. [2] Indeed, Birkhoff and von Neumann explained that the logic of experimental propositions as a calculus is different from classical logic but nevertheless "resembles the usual calculus of propositions with respect to and, or and not" [2]. They went on to study this system and analyse its difference with classical logic further, indicating that the distributive law had to make place for a weaker version, the so-called modular law. Later on, K. Husimi [14] showed that a further weakening of this law to the orthomodular law is necessary in the axiomatisation of the logic of projection operators on a Hilbert space.

In fact, von Neumann's main object was to provide a phenomenological basis at a very fundamental level for Hilbert space formations of quantum theory. In this connection he took physico-logical consideration [8, 10] for reducing contradictions and for shaping a theory better than the given theory. He discovered that the experimental propositions of a quantum system represented by the experimental propositions of a quantum mechanics represented by the set of all projections is a separable infinite dimensional complex Hilbert space (or equivalently by that of its all closed subspaces) was not a Boolean σ-logic (called the logic of classical system) but highly non-distributive orthocomplemented lattice. This pointed to the relevance to the theory of complemented lattices to the axiomatic foundation of quantum mechanics. In presence of modularity and finiteness of dimension, these lattices decomposed into direct sum of irreducible projective geometries. The theory of these geometries was further examined by Brikhoff and van Neumann [6] considering the two main observations : the Heisenberg uncertainty principle and Neumann's conclusion that two non-commutative observables could not be measured simultaneously, they made the central assumption that a projective geometry formed by the experimental propositions was a logic of quantum mechanics. The serious question that exists in their assumption was that they considered the existence of lattice join of two logical elements only due to technical necessities and not due to any intrinsic or inherent argument. It should be noticed that the collection of projections on a Hilbert space form a complete orthomodular atomic order symmetric lattice, which is not modular. But they did not abandon the idea that the lattice of quantum mechanical propositions was modular [5]. If their such idea is accepted, then it follows from the logic of quantum mechanics is a continuous geometry [27, 28]. Whether the logic of any atomic system may be assumed to be a continuous geometry is an open question. Moreover, every calculation in quantum mechanics was based on assumptions which not only implied that the set of experimental propositions was a logic but, in fact a very special one.

Mackey [15, 24] searched for a list of transparent and physically plausible axioms from which Hilbert space model could be deduced. Mackey's Axiom I-IV [16, p. 63-66] for quantum mechanics imply that the logic of quantum mechanical system should be a σ-orthocomplete orthomodular poset (i.e. an orthomodular poset, in which the supremum of every countable family of pairwise orthogonal elements exists). Mackey's work was further developed by C. Piron who justified the use of Hilbert space. Piron's theorem was later on improved by M. Solèr and R. Mayet [18, 22], who added one more axiom to the list to obtain a representation theorem with respect to the standard infinite-dimensional Hilbert space.

Further work in this area went in two main directions:

(1) a recasting of the Piron-Solèr-Mayet axioms in a propositional logic, and
(2) the quest of a similar representation theorem with respect to tensored Hilbert spaces describing compound physical systems.

Piron-Solèr-Mayet axioms can not be stated in the first-order language of orthomodular lattices [1] and secondly, some of their axioms were of rather un-intuitive character. Concerning the second direction, the research community obtained another impossibility results pointing to a weakness in the original quantum logic [6, 8]. The logic of quantum mechanics was investigated by several practitioners [11, 12, 13, 16, 17, 19, 20, 23, 24, 25 and others] and most of them reflected the theory at an agreement that a quantum logic being isomorphic to the partially ordered set of all projections of a separable, infinite dimensional complex Hilbert space is at least an orthomodular poset [11, 12, 13, 17, 19, 20, 24, 25 and others].

Randall and Foulis [21] introduced a new type of logic called an orthologic which consists of propositions associated with a general sample space, one in which the execution of more than one physical experiments is permissible. The logic was then investigated by Barbara [3, 4] who held it suitable for quantum mechanical formulism. Almost all logics associated with quantum mechanics are orthologic in which notion of compatibility can be introduced in the same way as introduced by Mackey [16, p. 70] and the notion of observables is the same way as introduced by Varadrajan [24, 25] in a logic. Our aim in this paper is to study to the logic of quantum mechanics.

Let L be a set of experimental propositions of a quantum mechanical system. This set is bounded partially ordered under the order $\leq$ defined as follow : if the truth value of a proposition b follows from that of a, than we say that a implies b and write $a \leq b$. A symmetric binary operation can be defined on L which satisfies the following axioms for all $a, b, c \in L$.

L_1 $a \perp a \Rightarrow a = 0$

L_2 $a \perp b \Rightarrow a \vee b$ exists in L

L_3 $a \perp b$, $a \perp c$ and $b \perp c \Rightarrow a \perp b \vee c$

L_4 $a \leq a$ holds iff $c \perp b \Rightarrow c \perp a$.

The following axioms which L satisfies reflects the notion of orthcomplementation.

L_5 for every $a \in L$, there exists $b \in L$ such that $a \perp b$ and $a \vee b = 1$.

The relation $a \perp b$ used in the above axioms is called the relation of orthogonality which can be interpreted to mean that a is true implies that b is false. The supremum $a \vee b$ (respectively infimum $a \wedge b$) has the usual meaning. In particular, if $a \perp b$, then it physically reasonable to write the supermum $a \vee b$ as $a + b$. It is clear from L_2 and L_3 that if $a_1, \ldots, a_n$, are pairwise orthogonal propositions, then $a_1 + a_2 + \ldots + a_n$ exists. The structure $(L, \leq, 0, 1, \perp)$ is called an orthologic. Having given a brief sketch of an orthologic of the set L of experimental propositions of quantum system, we can introduce the notion of state on L. A state of a quantum system is a map m on L with values in a closed unit interval satisfying the following axioms

S_1 $m(0) = 0$, $m(1) = 1$

S_2 if $\{a_i\}$ is a countable family of pairwise orthogonal propositions is L, then

$$m\left(\sum_{1}^{\infty} a_i\right) = \sum_{1}^{\infty} m(a_i)$$

If m_i is a sequence of states and $\lambda_i \geq 0$ such that $\sum_{1}^{\infty} \lambda_i = 1$, then $m = \sum_{1}^{\infty} \lambda_i m_i$ is a state. This state m is called a mixture of the m_i's and m is interpreted physically as being the state in which we only know that the system is in state m_i with probability λ_i. A state which is not a mixture is pure.

The concept of an observable associated with L is precisely the same as Varadrajan has associated its concept with a σ-logic, an observable is map x on a Borel subset $B(R)$ of the real line R with values in L satisfying the following axioms

O_1 $x(\phi) = 0$, $x(R) = 1$

O_2 $E, F \in B(R)$ and $E \cap F = \phi \Rightarrow x(E) \perp x(F)$

O_3 $E_i \in B(R)$ and $E_i \cap E_j = \phi$ for $i \neq j$

$$\Rightarrow x\left(\bigcup_{1}^{\infty} E_i\right) = \sum_{1}^{\infty} x(E_i).$$

The notion of compatibility or commutativity can be defined in L in the following way : two propositions a and b are said to be compitable or commutative (i.e. simultaneously verifiable) in symbol $a \leftrightarrow b$, if there exist mutually orthogonal propositions a_1, b_1 and d in L such that $a = a_1 + d$ and $b = b_1 + d$. If $A, B \subset L$, then $A \leftrightarrow B$ will mean that $a \leftrightarrow b$ for all $a \in A$ and for all $b \in B$. If x and y are two observables, then they are said to compitable (i.e. simultaneously measurable), in symbol $x \leftrightarrow y$, if for any two Borel subsets E and F of R

$$x(E) \leftrightarrow y(E).$$

The construction of generalised probability theory is, therefore, completed. Instead of being σ-algebra of subsets of a set our propositions, which are more general, form an orthologic with a less structure than σ-algebra. The probability measures are replaced by the states and the random variables by observables.

Definition 1.l. Let $a \in L$. Then $a' \in L$ is said to an orthogonal complement of a if for all $c \in L$, $c \perp a \Leftrightarrow c \leq a'$.

Definition 1.2. Let $a \in L$. Then a is said to be regular if $(L_{[0, a]}, \leq, 0, \perp)$ is an orthologic. If every element of L is regular, then L is said to a σ-orthologic.

Theorem 1.1. Every σ-logic (logic) [23] is σ-orthologic (orthologic) but the converse is not necessarily true.

Proof. It can be easily seen that every σ-logic is a σ-orthologic. Here we need only to prove the converse part of the theorem. Clearly an orthologic is a logic iff it is an orthocomplemented orthologic satisfying the following conditions : if $a \leq b$ then $b = a \vee (b \wedge a')$. But an orthocomplemented orthologic need not satisfy the above condition, hence need not be logic [4].

2. Center of an Orthologic

The center of quantum logic is defined in terms of its compatible elements. Here it will be shown that the center of an orthologic enjoys precisely the same property as that of an orthomodular poset.

Definition 2.1. The center of an orthologic L, denoted by $c(L)$ is the set of all a such that $\{a\} \leftrightarrow L$.

Theorem 2.1. The center of an orthologic L is a Boolean lattice.

The proof will be followed from a series of the following 9 lemmas concerning orthologic.

Lemma 2.1. For every $a \in L$, there exists a unique $b \in c(L)$ such that $a + b = 1$.

Proof. Let there be two propositions b and c such that $a + b = 1$ and $a + c = 1$. To show, the uniqueness of b, it is sufficient to show by symmetry of agreements $b \leq c$.

Suppose that there exists d in L such that $d \perp c$. Then since $a \in c(L)$ implies that $a \leftrightarrow b$ there exist mutually orthogonal propositions a_1, d_1 and e in L such that $a = a_1 + e$ and $d = d_1 + e$. Hence $d_1 \perp a_1 + e + c$. Thus $d_1 = 0$ and $d = e$. This means that $d \leq a$ and hence $a \perp b$ implies $d \perp b$. To show that $b \in c(L)$, we suppose that $c \in c(L)$ and show that $b \leftrightarrow c$. Since $a \in c(L)$, $a \leftrightarrow c$. Hence there exist mutually orthogonal propositions a_1 and c_1 and e such that $a = a + e$ and $c = c_1 + e$. Then $a + c_1 \in L$ implies that there exists f such that $a + c_1 + f = 1$. Hence $b = f + c_1$ (say) is unique. But $c = c_1 + e$ and f, c_1 and e are mutually orthogonal.

Lemma 2.2. Let $a \in L$ and b be a unique proposition in L such that $a + b = 1$. Then b is to be considered as an orthocomplementation of a in L.

Proof. It suffices to show that for every $c \in L$, $c \perp a$ implies that $c \leq b$. Now $c \perp a \Rightarrow c + a \in L$. Hence there exists d in L such that $c + a + d = 1$. Thus $b = c + a$ and hence $a \leq b$.

Lemma 2.3. Let $a \in c(L)$ and $b \in L$. Then $a \wedge b \in L$.

Proof. Since $a \leftrightarrow b$. Hence there exists mutually orthogonal propositions a_1, b_1, and e such that $a = a_1 + e$ and $b = b_1 + e$. It is clear that e is a lower bound for a and b. Assume there exists $c \in L$ such that $c \leq a$ and $c \leq b$. To prove that $a \wedge b = e$, it sufficies to show that for every $d \in L$, $d \perp e$ implies $d \perp c$. Since $a \in c(L)$, $a \leftrightarrow d$. Hence there exists mutually orthogonal propositions a_2, d_2 and f in L such that $a = a_2 + f$ and $d = d_2 + f$. Now f, e and b are mutually orthogonal and hence $f \perp b$. Since $c \leq b$, $f \perp b$ implies $f \perp e$. Moreover, since $c \leq a$, $d_2 \perp a$ implies $d_2 \perp c$. Hence d_2, f and c are mutually orthogonal propositions. Hence $d \perp e$.

Lemma 2.4. Let $a \in c(L)$, $b \in L$. Then $(a \wedge b') + (a \wedge b) = b$.

Proof. Since $a \leftrightarrow b$. Hence there exists mutually orthogonal propositions a_1, b_1, and e such that $a = a_1 + e$ and $b = b_1 + e$. Since $b_1 \perp a$ and $b_1 \leq a'$. Hence $b_1 \leq b \wedge a'$. Hence by Lemma 1.4.3 we have

$$(b \wedge a') + (b \wedge a) \geq b_1 + e = b.$$

Lemma 2.5. If for every $a \in L$, there exists a unique proposition $a' \in L$ such that $a + a' = 1$. Then L is an orthomodular poset.

Proof. It follows form lemma 2.2 that $(L, \leq, ')$ is an orthomodular poset if for every b such that $a \leq b$ we have $b = a \vee (b \wedge a')$. But $b \wedge a' = 0$. Hence it is enough to show that $a = b$. Now $a \leq b$ implies $a \perp b$ and hence $a + b' \in L$. Since $(b \wedge a')' = b' + a$ and $b \wedge a' = 0$. Hence $b' + a = 1$. But by hypotheses $a' + a = 1$ and a' is unique. Hence $a' = b'$, i.e. $a = b$.

Lemma 2.6. Let $a \in L$ and $b \in c(L)$ such that $a \leq b$. Then $a = (a + b') \wedge b$.

Proof. To show that a is the greatest lower bound for $a + b'$ and b suppose that there exists $c \in L$ such that $c \leq a + b'$ and $c \leq b$. Again suppose than there exists $d \in L$ such that $d \perp a$. Then it is enough to prove that $d \perp a$ implies $d \perp c$. Since $d \leftrightarrow b'$, there exist mutually orthogonal propositions d_1, b_1 and e in L such that $a = d_1 + e$ and $b_1' + e$. Since $d \perp a + b'$, $c \leq c + b'$ implies $d_1 \perp c$ and $e \leq b'$ implies $e \perp b$. Furthermore, since $c \leq L$ implies $e \perp c$. Hence $d \perp c$.

Lemma 2.7. Let $a, b \in L$, $e \in c(L)$ and $a \perp b$. Then

$$(a + b) \wedge c = (a \wedge c) + (b \wedge c)$$

Proof. From lemmas 2.4 and 2.6, we have

$$\begin{aligned}(a + b) \wedge c &= (a \wedge c) + (a \wedge c') + (b \wedge c) + (b \wedge c') \wedge c \\ &\leq [(a \wedge c) + (b \wedge c) + c'] \wedge c \\ &= (a \wedge c) + (b \wedge c)\end{aligned}$$

Since the remaining inequality is obvious, the lemma is completely established.

Lemma 2.8. Let $a, b \in c(L)$. Then $a \wedge c \in c(L)$.

Proof. Since $a \leftrightarrow b$, there exist mutually orthogonal propositions a_1, b_1 and e such that $a = a_1 + e$ and $b = b_1 + e$. By lemma 2.3, $a \wedge b = e$. Now c be any proposition in L. Then we shall show that $e \leftrightarrow c$. Since $a \leftrightarrow c$, and hence there exist mutually orthogonal propositions a_2, c_2 and f such that $a = a_2 + f$ and $c = c_2 + f$. Hence from lemmas 2.4 and 2.7, we have

$$\begin{aligned}c &= (c \wedge b') + (c \wedge b) \\ &= (c \wedge b') + (c_2 + f) \wedge b \\ &= (c \wedge b') + (c_2 \wedge b) \wedge (f \wedge b) \\ &= ((c \wedge b') + (c_2 \wedge b)) + (f \wedge b), \text{ and that} \\ e &= (a_2 + f) \wedge b \\ &= (a_2 \wedge b) + (f \wedge b)\end{aligned}$$

Hence it remains to prove that $a_2 \wedge b \perp (c \wedge b') + (c_2 \wedge b)$. This follows immediately from the fact that $b \perp b'$ and $a_2 \perp c_2$.

Lemma 2.9. If $a, b \in c(L)$, then $a \leftrightarrow b$ in $c(L)$.

Proof. Since $a \leftrightarrow b$, there exist mutually orthogonal propositions a_1, b_1 and e in L such that $a = a_1 + e$ and $b = b_1 + e$. By preceding lemma, $e \in c(L)$. Hence, by symmetry of arrangements, it is sufficient to show that $b_1 \in c(L)$. We have

$$\begin{aligned}b \wedge a' &= (b_1 + e) \wedge a' \\ &= (b_1 \wedge a')\ (e' \wedge a')\end{aligned}$$

But $b_1 \perp a$ implies $b_1 \wedge a' = b_1$ and $e \leq a$ implies $e \wedge a' = 0$. Hence $b_1 = b \wedge a'$. But $b \wedge a'$ is a central proposition, since $b, a' \in c(L)$. Hence $b \in c(L)$.

Proof of the main theorem. It is easy to see that $(c(L), \leq, 0, 1, \perp)$ is an orthologic. By lemma 2.1 and 2.5 it is orthormoduler poset. By lemma 2.8, $a, b \in c(L)$ implies $a \wedge b \in c(L)$. Since $c(L)$ is orthocomplemented, it follows from de Morgan's law that $a \vee b \in c(L)$. Hence $c(L)$ is an orthomodular lattice. Moreover, it has been shown in lemma 2.9 that $a \leftrightarrow b$ for every $a, b \in c(L)$. Hence it is a Boolean lattice [25, p. 127].

It has been seen in the previous theorem that a Boolean lattice can also be characterised as an orthologic in which each of its elements is compitable with each of its other elements. With this concept it will be seen that an important theorem on simultaneous observables in a σ-logic [20, 24] is also true in a σ-orthologic.

Theorem 2.2. If x and y are two compitable observables in σ-orthologic L, then there exists a observable z and Borel functions u and v satisfying the conditions. $x = z \circ u^{-1}$ and $y = z \circ v^{-1}$ [38].

Proof. Let X and Y respectively denote the images of x and y. It is easy to see that X and Y are Boolean sub σ-logics of L. Since by hypothesis $X \leftrightarrow Y$. Hence it follows from theorem 2.1, that there exists a smallest Boolean sublogic Z of L containing $X \cup Y$. Since X and Y are both σ-homomorphisms, X and Y are countably generated as Boolean σ-lattice [23, proposition 3]. If Z_x and Z_y are their respective countable generating sets, then

$$\begin{aligned}X \cup Y &= [Z_x] \cup [Z_y] \\ &\subseteq [Z_x \cup Z_y] \\ \text{i.e. } [X \cup Y] &= Z\end{aligned}$$

where $[Z_x]$ denotes the smallest Boolean sublogic of L containing Z_x and similar meaning can be given to the others. Hence

$$[X \cup Y] = [Z_x \cup Z_y] = Z$$

So $Z_x \bigcup Z_y$ is the countable generating set for *Z*. Hence there exists a σ-homomorphism

$$Z : B(R) \to Z$$

which is an observable [24]. The rest part of the proof of the theorem follows from [20].

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