A RELATIVISTIC MODEL FOR ANISOTROPIC FLUID SPHERE

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Abstract

Exact analytical solutions of Einstein's field equations are of much value in general relativity. These solutions are generally obtained by using different conditions and assumptions. Here in this paper an attempt has been made to obtained an exact analytical solution of Einstein's field equations for static anisotropic fluid sphere by assuming that space time is conformally flat and by taking a suitable form of metric potential g_{11}. The model is physically reasonable and free from singularities. Energy density ρ, radial and tangency pressures have been calculated for the model.

Keywords : Einstein Field equation, General theory of relativity, anisotropic, fluid sphere

1. INTRODUCTION

Many research workers in theory of relativity have focussed their mind towards finding solution of Einstein's Field Equations. In fact various theories of gravitation have been proposed, but it is generally accepted that the most successful is Einstein's Theory of General Relativity. The well known Schwarzschild interior solution representing the field of a fluid sphere of constant density ρ, was discovered more than ninety years ago and still holds a prominent place in relativity theory. Later on, Tolman[22] expressed Einstein's field equation is a mathematically convenient form and making suitable assumptions on the metric co-efficients, obtained a number of interesting solutions including the solution for the Einstein universe, the Schwarzschild, De-Sitter solution and the Schwarzschild interior solution.

In fact, exact solutions to the Einstein's field equations in closed analytic form are difficult to obtain due to high non-linearity of the equations. So, a small number of exact solutions have been obtained. The problem of constructing a static modern sphere of perfect fluid (e.g. neutron model) is usually solved by numerical methods using Tolman-Openheimer Volkoff [17], [18], [22] equation with an equation of state specified. A small number of analytic solutions which have been obtained are valuable and interesting because one may study their properties in complete details and with comparative case, specially their behaviour at high field intensity or high pressure and density. The analytic solutions are thus complimentary to the numerical solutions obtained with realistic equation of state. Mehra et al [16] have obtained a general solution of the field equations for a composite sphere having a number of shells of different densities. Durgapal and Gehlot [7] have obtained exact internal solutions for dense massive stars in which the central pressure and density are infinitely large. Durgapal and Gehlot [8,9] have further obtained exact solutions for a massive sphere with two different density distribution. The density being minimum at the surface varies inversely as the square of distance from the centre. The distribution has a core of constant density and radius. Solutions of Einstein's field equations for spherically symmetric matter distribution have been also solved by Adler [1], Whitman [23], Singh and Yadav [20] and Yadav and Saini [24] using different methods and assumptions. The matter distribution is usually assumed to be locally isotropic, in case of pressure for massive objects in general relativity. However, in the last few years, theoretical studies on relativistic stellar models indicate that some massive objects may be locally anisotropic, [2,5,15]. There are a number of interesting solutions that have provided insight into the effects of anisotropy on star parameters [6,12,13]. However, many of these solutions have a limited applicability to astrophysical situations. Since they do not satisfy certain physical restrictions usually imposed upon density and pressure, viz; that the pressure should not exceed the energy density (dominant energy condition) and that the (adiabatic) derivatives of the pressure with respect to the density should be less than or equal to unity [11] (macro causality condition).

Exact analytical solutions of Einstein's field equations are of much value in general relativity. These solutions are generally obtained by using different conditions and assumptions. One of the assumptions made for obtaining the solutions is that the space time be conformally flat. This assumption has been widely used in relativity theory [3,4,10,14,21].

Here in this paper an attempt has been made to obtained an exact analytical solution of Einstein's field equations for static anisotropic fluid sphere by assuming that space time is conformally flat and by taking a suitable form of metric potential g_{11}. The model is physically reasonable and free from singularities. Energy density ρ, radial and tangency pressures have been calculated for the model.

2. THE FIELD EQUATIONS

We consider the static spherically symmetric line element in the form given by

(4.2.1) $ds^2 = e^{\beta}dt^2 - e^{\alpha}dr^2 - r^2\,(d\theta^2 + \sin^2\,\theta\, d\phi^2\,)$

Where α and β are functions of r only.

The Einstein's field equations

(4.2.2) $R^i_j - \frac{1}{2}R\,\delta^i_j\, = \,-8\,\pi T^i_j$

For the spherically line element, (4.2.1) gives

(4.2.3) $8\,\pi T^1_1 = e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$

(4.2.4) $-8\,\pi T^2_2 = -8\,\pi T^3_3 = e^{-\alpha}\left(\frac{\beta''}{2} + \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} + \frac{\beta'-\alpha'}{2r}\right)$

(4.2.5) $8\,\pi T^4_4 = e^{-\alpha}\left(\frac{\alpha'}{4} - \frac{1}{r^2}\right) + \frac{1}{r^2}$

where a prime denotes differentiation with respect to r.

Throughout the investigation, we set velocity of light c and gravitational constant k to be unity. The energy momentum tensor T_j^i is given by

(4.2.6) $T_j^i = (\rho + p)u^i u_j - p\delta_j^i$

For anisotropic fluid sphere, the field equations given above go to the form

(4.2.7) $-8\pi\rho = e^{-\alpha}\left(\frac{1}{r^2} - \frac{\alpha'}{r}\right) - \frac{1}{r^2}$

(4.2.8) $-8\pi p_r = \frac{1}{r^2} - e^{-\alpha}\left(\frac{1}{r^2} + \frac{\alpha'}{r}\right)$

(4.2.9) $-8\pi p_\perp = e^{-i}\left[\frac{\beta'\alpha'}{4} - \frac{\beta'^2}{4} - \frac{\beta''}{2} - \frac{\beta'-\alpha'}{2r}\right]$

where ρ is energy density and p_r and $p_\perp$ are the radial and tangential pressure respectively.

The non-zero components of Weyl tensor for the metric (4.2.1) are

(4.2.10) $C_{1212} = \frac{r}{12}(\beta' + \alpha' + r\beta'') - \frac{1}{6}(e^\alpha - 1) - \frac{r^2}{24}(\beta'\alpha' + \beta'^2)$

$C_{1313} = \sin^2\theta\, C_{1212}$

$C_{1010} = \frac{2e^\beta}{r^2} C_{1212}$

$C_{2323} = -2\sin^2\theta\, e^{-\alpha} r^2 C_{1212}$

$C_{2020} = e^{\beta-\alpha} C_{1212}$

$C_{3030} = -\sin^2\theta\, e^{\beta-\alpha} C_{1212}$

We suppose that the space time is conformally flat for which vanishing of Weyl tensor gives

(4.2.11) $\frac{e^\alpha}{r^2} + \frac{\alpha'\beta'}{4} - \frac{1}{r^2} - \frac{\beta'^2}{4} - \frac{\beta''}{2} + \frac{\beta'-\alpha'}{2r} = 0$

Now we make use of transformations

(4.2.12) $\zeta = e^{-\lambda}$

(4.2.13) $\eta^2 = e^\nu$

(4.2.14) $x = r^2$

(4.2.15) $x\zeta,x + 1 - \zeta - 4\pi x A = 0$

(4.2.16) $(4\zeta x^2)\eta,xx + (2x^2\zeta,x - \zeta + 1)\eta = 0$

where $A = p_r - p_\perp$ and the subscript x following a comma denotes differentiation w.r.t. x.

Equations (4.2.15) and (4.2.16) on integration yields

(4.2.17) $\zeta = e^{-\lambda} = 1 + \lambda\mu r^2 + 8\pi r^2 \int_o^r (p_r - p_\perp)/r \; dr$

(4.2.18) $\eta^2 = e^{\nu} = r^2\left[Ae^{H(r)} + Be^{-H(r)}\right]^2$

where μ, A and B are constants of integration and

(4.2.19) $H(r) = \int (e^{\frac{\lambda}{2}}/r)dr$

The integration constants μ, A and B can be evaluated by matching the metric functions given by (4.2.17) and (4.2.18) to the exterior Schwarzschild solution for a mass m and radius r_0 as

(4.2.20) $A = \frac{e^{\frac{H(r_0)}{2}}}{2r_0}\left[\left(1 - \frac{2m}{r_0}\right)^{\frac{3}{2}} + \frac{3m}{r_0} - 1\right]$

(4.2.21) $B = \frac{e^{\frac{H(r_0)}{2}}}{2r_0}\left[\left(1 - \frac{2m}{r_0}\right)^{\frac{3}{2}} - \frac{3m}{r_0} + 1\right]$

(4.2.22) $e^{\lambda(r_0)} = 1 - \frac{2m}{r_0}$

3. SOLUTIONS OF THE FIELD EQUATIONS

We see that as a matter of fact equations (4.2.7)-(4.2.9) and (4.2.17)-(4.2.19) are three equations in four unknowns $\rho, p_r, p_\perp$ and $H(r)$. Thus the system is indeterminate. To make the system determinate, we choose

(4.3.1) $\sqrt{\zeta} = e^{-\frac{\lambda}{2}} = \frac{1+Dr^2}{1-Er^2}$

where D and E are constants. $e^{\nu}, p_r, p_\perp$ and ρ can be found from field equations and using (4.2.16)-(4.2.22). However, to simplify mathematics we take $D = \mu = \frac{E}{3}$, which yields the solution as

(4.3.2) $e^{\frac{\lambda}{2}} = \frac{1+3G\epsilon}{1-G\epsilon} = 1 + \frac{4G\epsilon}{1-G\epsilon}$

(4.3.3) $\square^{\frac{\square}{2}} = \frac{4\square\epsilon(1-\square)^4+(1+\square)(1-3\square)(1-\square\epsilon)^4}{[(1-\square)(1+3\square)(1-\square\epsilon)]^2}$

(4.3.4) $8\square\square\square_0^2 = \frac{8\square(3+2\square\epsilon+3\square^2\epsilon^2)}{(1+3\square\epsilon)}$

(4.3.5) $8\square\square_{\square}\square_0^2 = \frac{16\square\left[\left\{\left(\frac{(1-\square)^4}{(1-3\square)(1+\square)-1}\right)+1\right\}(1-2\square\epsilon+3\square^2\epsilon^2)-(1-\square\epsilon)^4\right]}{\left[4\square\left\{\left(\frac{(1-\square)^4}{(1-3\square)(1+\square)-1}\right)+1\right\}\epsilon+(1-\square\epsilon)^4\right](1+3\square\epsilon)^4}$

$G = \frac{1-\left(1-2\frac{m}{r_0}\right)^{\frac{1}{2}}}{1+3\left(1-2\frac{m}{r_0}\right)^{\frac{1}{2}}}$

(4.3.6) $8\square\square_{\perp}\square_0^2 = 8\square\square_{\square}\square_0^2 + \frac{16\square^2\epsilon(5+3\square\epsilon)}{(1-3\square\epsilon)^3}$

where $\epsilon = \frac{\square^2}{\square_0^2}$

4. RESULTS and DISCUSSION

The main highlight of this work is as follows:

1. It is remarkable that the solution of Einstein's field equations obtained here is singularity free and the density of fluid sphere drops continuously from its maximum value at the centre to the value which is positive at the boundary.
2. If we choose the equation of state $p_r = p_{\perp}$, then in this case we obtain the well known Schwarzschild interior solution [19].

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