Performance Ranking using Type-2 Fuzzy DEA: An Application to Indian Mutual Fund Industry

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Abstract: The huge growth of mutual fund (MF) industries paying attention to individual institutions to invest in mutual funds. This become an important instrument to gain wealth with less risk. Thus to get an idea about an MF one has to go through the performance of the MFs. The measurement of performance of asset management company (AMC) and a particular mutual fund is not simple matter, it depends upon several criteria which are both quantitative and qualitative in nature. These variables contain a large amount of uncertainties. Type 1 and type 2 fuzzy sets can be used to meet uncertainties. One of the key measures to analyse the performance of MFs is ranking of MFs. Type 2 fuzzy sets with critical reduction techniques developed for credibility value-at-risk (VaR) which maps a fuzzy credibility space to the real number space. I have introduced fuzzy data envelopment analysis in type 2 fuzzy environment and model is designed with inputs and outputs, which can be solved numerically using general purpose software (LINGO 14.0). Finally, a case study is presented to rank a number of MFs from Indian Asset Management Companies (AMC) on the basis of their efficiencies using proposed DEA-VaR reduction model.

Index Terms **- Mutual Funds, Data Envelopment Analysis, Type-2 Fuzzy Variable, value-at-risk (VaR), Critical value, Performance Evaluation**

I. INTRODUCTION

Over the years' mutual funds in various sectors providing good return with less risk. Mutual funds in a sector designed with a portfolio carrying stocks of several companies with majority from a particular sector. For example, a mutual fund in banking sector contains majority of bank stocks. They are engaged in buying and selling stocks in trading time. The net asset value (NAV) is calculated based on the stock price present in the portfolio in every trading day. Portfolio theory was developed mathematically with the pioneering work from Markowitz [18,19], according to which portfolio selection problem can be formulated on maximizing return and minimizing risk, on the basis of historical data of the stock(s). Several methods are developed, for example, the CAMP, capital assets pricing model, (Sharpe [31], Linter [15], Mossins [21]) and the arbitrage pricing theory (APT) (Ross, [28]).

The evaluation of performance of mutual funds plays a vital role in selecting a particular MF from several schemes. Treynar [32], Sharpe [31], Jensen [7,8,9] proposed MF portfolio performance and/or ranking under risk aversion models. Pendaraki [24] considers in evaluation of MF portfolio using data envelopment analysis (DEA) considering higher moment on efficiency of a MF portfolio. Zadeh [34] extended type-1 fuzzy set to type 2 fuzzy set. The type 2 fuzzy sets have been used efficiently in performance analysis to manage uncertainties in variables. The concept of embedded type-1 fuzzy number is used for ranking type-2 fuzzy number (Mitchell [20]). In type 2 fuzzy set the membership function for each member of the variable is a type 1 fuzzy set. There are several standard methods to defuzzyfy type 1 fuzzy set to crisp value. Then reduction of type 2 fuzzy set become essential.

Karnik et al. [10,11] developed a reduction method for type 2 fuzzy set into type 1 fuzzy set using defuzzification method with centroid measurement and subsequently this method developed by Liu [17]. Interval valued membership function for type 2 fuzzy set was considered and developed a probability type reduction by Qin et al. [27]. Qin et al. [26] proposed three kinds CVs (critical values) for T2FS in triangular form. Kundu et al. [12] used T2FS variables in triangular form and used critical value reduction to solve transportation problem. A multi objective DEA type 2 fuzzy model was considered by Zhou et al. [35] to evaluate both efficiency effectiveness of integrated productivity of sustainable suppliers. Interval T2FS was used by Wu and Mendel (2007) in some uncertainty measures like variance, skewness and randomness (entropy).

To measure efficiency performance of MFs comparing with other peer MFs, DEA (Charnes et al. [3]) was used by Murthi et al. [23] through judgements. Hajiagha et al. [6] extended DEA model in which inputs and outputs are intuitionistic fuzzy sets. They applied this method for efficiency evaluation under uncertainty environment of financial and credit institution. Abdollahi et al. [1] used fuzzy DEA to rank supplier based on ANP which determines weights of selected criteria. Langroudi et al. [13] proposed a fuzzy DEA method for computing the ideal values and distance function of type – 2 attributes in TOPSIS methodology.

This paper is designed to evaluate performance of a group of mutual funds by considering some parameters like inputs: Asset under Management (AUM), Expense Ratio, one year CNX Nifty Benchmark Return, 1 year Fund Return; and outputs: Mean Return, Standard Deviation, Sharpe Ratio, Information Ratio, using DEA models with parameter are taken as type-2 fuzzy variable. We have used critical value formula of reduced fuzzy variable through the possibility value-at-risk reduction method Bai et al. [2].

The model is organized follows. Section 2 reviews the existing approaches to measuring mutual fund performance, the review of DEA, and discussion of type-2 fuzzy set reduction for triangular type-2 fuzzy variable. Section 3 discusses the proposed type-2 Fuzzy DEA model with VaR reduction method. An illustrative example is produced to illustrate our model in Section 4. The results of case study on Indian Mutual Funds are presented in Section 5, while Section 6 concludes.

II. MATHEMATICAL PRELIMINARIES

A. Possibility Measure

Let $\mathcal{P}(\Omega)$ be the power set of Ω. A possibilistic measure is a function Pos: $\mathcal{P}(\Omega) \rightarrow [0,1]$ satisfying following axioms:
Axiom-1 : Pos(ϕ)=0, $Pos(\Omega)=1$, ϕ is the null set.
Axiom-2: For every $\mathcal{X}, \mathcal{Y}$ in $\mathcal{P}(\Omega)$ and $\mathcal{X} \subseteq \mathcal{Y}$, then Pos($\mathcal{X}$)≤Pos($\mathcal{Y}$)
Axiom-3: For any number of subsets ranging in an index set $\mathcal{I}$, $\{\mathcal{X}_i \in \mathcal{P}(\Omega) : i \in \mathcal{I}\}$, $\mathrm{Pos}\left(\bigcup_{i \in I} X_i\right) = \sup_{i \in I} \mathrm{Pos}(X_i)$ (9)

It can be uniquely determined by a possibility distribution function f: $\Omega \rightarrow [0,1]$ by $Pos(A) = \sup_{x \in A} f(x), A \subseteq \Omega$.

Example 1 (Qin . et al., [26])
Let Ω={0,1,2,3,4,5,6,7,8,9,10}. Let us define
Pos({x}):=Possibility that x is close to 8 (Table I).

Table I: Possibilty value of different members of Ω

x	0	1	2	3	4	5	6	7	8	9	10
Pos	0.0	0.0	0.0	0.0	0.0	0.1	0.5	0.8	1	0.8	0.5

Pos($\mathcal{X}$):=Possibility that the event $\mathcal{X}$ contains all integers close to 8.
For $\mathcal{X}$={2,5,9}, $Pos(\mathcal{X})$=Max{0.0, 0.1,0.8}=0.8.

B. Critical Value Measure for Triangular Type-2 variable

A type-2 fuzzy variable $\tilde{\xi}$ is called type-2 triangular fuzzy variable (T2TFV) if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is given by

$$\left(\frac{x-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}, \frac{x-r_1}{r_2-r_1}, \frac{x-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}\right), r_1 \le x \le r_2$$

$$\left(\frac{r_3-x}{r_3-r_2} - \theta_l \min\left\{\frac{x-r_2}{r_3-r_2}, \frac{r_3-x}{r_3-r_2}\right\}, \frac{r_3-x}{r_3-r_2}, \frac{r_3-x}{r_3-r_2} + \theta_r \min\left\{\frac{x-r_2}{r_3-r_2}, \frac{r_3-x}{r_3-r_2}\right\}\right), r_2 \le x \le r_3 \cdot$$

where θ_l, $\theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\xi}$ is taking at x. We denote this fuzzy set by ($r_1,r_2,r_3;\theta_l,\theta_r$). Let θ=(θ_l, θ_r). Then the possibility distribution of $\tilde{\xi}$ is given by

$$\mu_{\tilde{\xi}^U}(x,\theta,\alpha) = Sup\{x \mid Pos\{\xi \ge x\} \ge \alpha\}$$

$$= \begin{cases} \{1+(1-\alpha)\theta_r\}\dfrac{x-r_1}{r_2-r_1}, & if\ x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{x-r_1}{r_2-r_1} + (1-\alpha)\theta_r \dfrac{r_2-x}{r_2-r_1}, & if\ x \in \left[\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{r_3-x}{r_3-r_2} + (1-\alpha)\theta_r \dfrac{x-r_2}{r_3-r_2}, & if\ x \in \left[r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{r_3-x}{r_3-r_2} + (1-\alpha)\theta_r \dfrac{r_3-x}{r_3-r_2}, & if\ x \in \left[\dfrac{r_2+r_3}{2}, r_3\right] \end{cases}$$

$$\mu_{\tilde{\xi}^L}(x,\theta,\alpha) = Sup\{x \mid Pos\{\xi \ge x\} \ge \alpha\}$$

$$= \begin{cases} \dfrac{x-r_1}{r_2-r_1} - (1-\alpha)\theta_l \dfrac{x-r_1}{r_2-r_1}, & if\ x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{x-r_1}{r_2-r_1} - (1-\alpha)\theta_l \dfrac{r_2-x}{r_2-r_1}, & if\ x \in \left[\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{r_3-x}{r_3-r_2} - (1-\alpha)\theta_l \dfrac{x-r_2}{r_3-r_2}, & if\ x \in \left[r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{r_3-x}{r_3-r_2} - (1-\alpha)\theta_l \dfrac{r_3-x}{r_3-r_2}, & if\ x \in \left[\dfrac{r_2+r_3}{2}, r_3\right] \end{cases}$$

Definition 1. Possibilistic Value at Risk (VaR)
Let $\tilde{\xi}$ be a RFV. The VaR (upper) and VaR (lower) of $\tilde{\xi}$ relative to the possibility distribution are respectively defined by for a given $\alpha \in (0,1)$,

$$\left.\begin{aligned} VaR_\alpha^U(\tilde{\xi}) &= Sup\{x \mid pos\{\tilde{\xi} \ge x\} \ge \alpha\} \\ VaR_\alpha^L(\tilde{\xi}) &= \inf\{x \mid pos\{\tilde{\xi} \le x\} \ge \alpha\} \end{aligned}\right\}$$

Theorem 1 (Bai et al. [2])
Let $\tilde{\xi}$ =($r_1,r_2,r_3;\theta_l,\theta_r$) be a T2TFV. Let θ=(θ_l, θ_r). Then the possibility distribution of upper critical value of the upper reduced fuzzy variable $\tilde{\xi}^U$ is given by:

$$\tilde{\xi}^{U}_{\inf}(\beta;\theta,\alpha)=\begin{cases} r_1+\dfrac{2\beta(r_2-r_1)}{1+(1-\alpha)\theta_r}, & \text{if } \beta\in\left[0,\dfrac{1+(1-\alpha)\theta_r}{4}\right] \\ r_2-\dfrac{(1-2\beta)(r_2-r_1)}{1-(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{1+(1-\alpha)\theta_r}{4},\dfrac{1}{2}\right] \\ r_2+\dfrac{(2\beta-1)(r_3-r_2)}{1-(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{1}{2},\dfrac{3-(1-\alpha)\theta_r}{4}\right] \\ r_3-\dfrac{2(1-\beta)(r_3-r_2)}{1+(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{3-(1-\alpha)\theta_r}{4},1\right] \end{cases}$$

$$\tilde{\xi}^{U}_{\sup}(\beta;\theta,\alpha)=\begin{cases} r_3-\dfrac{2\beta(r_3-r_2)}{1+(1-\alpha)\theta_r}, & \text{if } \beta\in\left[0,\dfrac{1+(1-\alpha)\theta_r}{4}\right] \\ r_2+\dfrac{(1-2\beta)(r_3-r_2)}{1-(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{1+(1-\alpha)\theta_r}{4},\dfrac{1}{2}\right] \\ r_2-\dfrac{(2\beta-1)(r_2-r_1)}{1-(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{1}{2},\dfrac{3-(1-\alpha)\theta_r}{4}\right] \\ r_1+\dfrac{2(1-\beta)(r_2-r_1)}{1+(1-\alpha)\theta_r}, & \text{if } \beta\in\left[\dfrac{3-(1-\alpha)\theta_r}{4},1\right] \end{cases}$$

Theorem 2 (Bai et al. [2]

Let $\tilde{\xi}=(r_1,r_2,r_3;\theta_l,\theta_r)$ be a T2TFV. Let $\theta=(\theta_l,\theta_r)$. Then the possibility distribution of lower critical value of the lower reduced fuzzy variable $\tilde{\xi}^{L}$ is given by:

$$\tilde{\xi}^{L}_{\inf}(\beta;\theta,\alpha)=\begin{cases} r_1+\dfrac{2\beta(r_2-r_1)}{1-(1-\alpha)\theta_l}, & \text{if } \beta\in\left[0,\dfrac{1+(1-\alpha)\theta_l}{4}\right] \\ r_2-\dfrac{(1-2\beta)(r_2-r_1)}{1+(1-\alpha)\theta_l}, & \text{if } \beta\in\left[\dfrac{1+(1-\alpha)\theta_l}{4},\dfrac{1}{2}\right] \\ r_2+\dfrac{(2\beta-1)(r_3-r_2)}{1+(1-\alpha)\theta_{l_r}}, & \text{if } \beta\in\left[\dfrac{1}{2},\dfrac{3+(1-\alpha)\theta_l}{4}\right] \\ r_3-\dfrac{2(1-\beta)(r_3-r_2)}{1-(1-\alpha)\theta_l}, & \text{if } \beta\in\left[\dfrac{3+(1-\alpha)\theta_l}{4},1\right] \end{cases}$$

$$\tilde{\xi}^{L}_{\sup}(\beta;\theta,\alpha)=\begin{cases} r_3-\dfrac{2\beta(r_3-r_2)}{1-(1-\alpha)\theta_l}, & \text{if } \beta\in\left[0,\dfrac{1-(1-\alpha)\theta_l}{4}\right] \\ r_2+\dfrac{(1-2\beta)(r_3-r_2)}{1+(1-\alpha)\theta_l}, & \text{if } \beta\in\left[\dfrac{1+(1-\alpha)\theta_l}{4},\dfrac{1}{2}\right] \\ r_2-\dfrac{(2\beta-1)(r_2-r_1)}{1+(1-\alpha)\theta_l}, & \text{if } \beta\in\left[\dfrac{1}{2},\dfrac{3+(1-\alpha)\theta_l}{4}\right] \\ r_1+\dfrac{2(1-\beta)(r_2-r_1)}{1+(1-\alpha)\theta_l}, & \text{if } \beta\in\left[\dfrac{3+(1-\alpha)\theta_l}{4},1\right] \end{cases}$$

C. Data envelopment analysis (DEA)

DEA is a non-parametric method to find efficiency of a firm or a decision making unit (DMU). The method DEA was developed by Charnes, Cooper and Rhodes [3], also known as CCR model. Efficiency of a DMU in case of a single input and output is measured by the ratio of output over input. It is a programing approach that evaluates a group of DMU with comparative efficiency. The purposes of DEA are (i) identifying the best alternatives, (ii) ranking alternatives, (iii) establishing a short list of the better alternatives for detailed review (Cook et al. [4]).. The performance of a DMU is measured with respect to multiple inputs and outputs as

$$Performance=\frac{weighted\ \ sum\ of\ \ outputs}{weighted\ \ sum\ of\ \ inputs} \qquad)$$

The problem of assigning weights to inputs and outputs has been one of the most challenging problems in measuring efficiency. The major contribution of DEA is that it derives weights parameter (u_i,v_j) from the data itself and so it does not require predetermined weights for calculating the efficiency of DMU. That is why it is called a non-parametric method.

Let us suppose that there are n DMU with common m inputs and s outputs. We are given: x_{ij}=amount of input I used by DMUj. (i=1,2, …, m; j=1,2, …, n) ;

y_{rj}=amount of output r (r=1,2, …, s), generated by DMU j.

m=number of inputs, *n*=number of outputs.

Let us assume: θ_j = efficiency rating of DMU j, u_i = weight or coefficient assigned by DEA to input I, v_j = weight or coefficient assigned by DEA to output r.

To find the coefficients u_i and v_r we use the linear programming technique. For DMU j the CCR DEA model can be formulated as:

$$Max\theta_j=\frac{v_1y_{1j}+v_2y_{2j}+\cdots+v_ny_{nj}}{u_1x_{1j}+u_2x_{2j}+\cdots+u_mx_{mj}}=\frac{vy_j^T}{ux_j^T}$$

Subject to the constraints:

$$\frac{vy_j^T}{ux_j^T}=\frac{v_1y_{1j}+v_2y_{2j}+\cdots+v_ny_{nj}}{u_1x_{1j}+u_2x_{2j}+\cdots+u_mx_{mj}}\le 1 \text{ for DMU}_j\, j=1,2,\ \ldots\ n.$$

$$u_1\ge 0,u_2\ge 0,\cdots,u_m\ge 0,v_1\ge 0,v_2\ge 0,\cdots v_n\ge 0\,.$$

v=$(v_1,v_2, \ldots, v_s)_{1\times s}$; **y**$_j$=$(y_{1j},y_{2j}, \ldots ,y_{sj})_{1\times s}$; **u**=$(u_1,u_2, \ldots, u_m)_{1\times m}$; **x**$_j$=$(x_{1j},x_{2j}, \ldots, x_{mj})_{1\times m}$

The above problem is not an LPP. A simple work around is to fix the denominator a constant value e.g. 1.0, which can be interpreted as setting a constant on weights ui (often weights are normalized). This results in:

Max. $\theta_j=\mathbf{v}\,\mathbf{y}_j^{\mathbf{T}}$

Subject to

$ux_j^T = 1$ for $j=1,2, \ldots n.$

$vy_j^T - ux_j^T \le 1$, for $j=1,2, \ldots n$

$u \ge 0, v \ge 0$

The relative efficiency of mutual fund compared to a set of other mutual funds can be assessed by data envelopment analysis. Charnes, Cooper and Rhodes (CCR) 1978 introduced the dual of the above problem as:

$$\left.\begin{array}{l} Min. \quad \theta_j - \left(\varepsilon \sum s_r^+ + \varepsilon \sum s_i^-\right) \\ x_{ij} - s_i^- - \sum_{j=1}^{n} x_{ij}\lambda_j = 0, \quad i = 1,2,\ldots,m \\ -s_r^+ + \sum_{j=1}^{n} y_{rj}\lambda_j = 0, \quad r = 1,2,\ldots,m \\ \lambda_j \ge 0,\ s_i^- \ge 0,\ s_r^+ \ge 0, \theta_j \ unrestricted\ in\, sign. \end{array}\right\}$$

D. Fuzzy DEA

Fuzzy set theory has been used in DEA to combat imprecise inputs and outputs of DMUs. The combination of fuzzy set and DEA led the idea of fuzzy data envelopment analysis (FDEA). Saati *et al.* [29], Puri *et al.* [25], Guo *et al.* [5], Leon *et al.* [14] applied FDEA to find efficiency and rank several DMUs. The possibility measure of fuzzy set approach applied to transform a non-linear FDEA into a linear possibility DEA model. A DEA model for j-th DMU is

$$\left.\begin{array}{l} \max\limits_{u,v} \tilde{\theta}_j = \dfrac{v\,\tilde{y}_j^T}{u\,\tilde{x}_j^T} \\ \text{subject to } \dfrac{v\,\tilde{y}_j^T}{u\,\tilde{x}_j^T} \le 1 \\ u \ge 0, v \ge 0 \\ j = 1,2,\ldots,n \end{array}\right\}$$

where $\tilde{x}_j = \left(\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj}\right)$, the m-dimensional type-1 fuzzy input vector for jth DMU (j=1,2,…,n), $\tilde{y}_j = \left(\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{sj}\right)$, the s-dimensional type-1 fuzzy output vector for jth DMU (j=1,2, ….,n).

III. PROPOSED TYPE-2 FUZZY DEA METHODOLOGY UNDER VAR REDUCTION

The fuzzy DEA model in type- 2 fuzzy variable can be written as

$$\left.\begin{array}{l} \max\limits_{u,v} \tilde{\tilde{\theta}}_j = \dfrac{v\,\tilde{\tilde{y}}_j^T}{u\,\tilde{\tilde{x}}_j^T} \\ subject\ to\ \dfrac{v\,\tilde{\tilde{y}}_j^T}{u\,\tilde{\tilde{x}}_j^T} \le 1 \\ u \ge 0, v \ge 0 \\ j = 1,2,\ldots,n \end{array}\right\}$$

where

$\tilde{\tilde{x}}_j = \left(\tilde{\tilde{x}}_{1j}, \tilde{\tilde{x}}_{2j}, \ldots, \tilde{\tilde{x}}_{mj}\right)$, the *m*-dimensional type-2 fuzzy input vector for *j* th DMU (*j=1,2,…,n*)

$\tilde{\tilde{y}}_j = \left(\tilde{\tilde{y}}_{1j}, \tilde{\tilde{y}}_{2j}, \ldots, \tilde{\tilde{y}}_{sj}\right)$, the *s*-dimensional type-2 fuzzy output vector for *j* th DMU (*j=1,2,…,n*).

We now apply the proposed possibility value at risk (VaR) reduction method to obtain .

$$\left.\begin{array}{l} \max\limits_{u,v} \tilde{\tilde{f}} \\ \text{subject to} \\ \tilde{C}r\left\{\dfrac{v\,\tilde{\tilde{y}}_j^T}{u\,\tilde{\tilde{x}}_j^T} \ge \tilde{\tilde{f}}\right\} \ge \alpha \\ \tilde{C}r\left\{v\,\tilde{\tilde{y}}_j^T - u\,\tilde{\tilde{x}}_j^T \le 0\right\} \ge \alpha_j \\ u \ge 0, v \ge 0 \\ j = 1,2,\ldots,n \end{array}\right\}$$

Let us suppose that $\tilde{\tilde{x}}_j$, $\tilde{\tilde{y}}_j$ for j th DMU ($j=1,2,\ldots,n$) are mutually independent type-2 triangular fuzzy vectors with elements

$$\tilde{\tilde{x}}_{ij}=\left(a_{\tilde{\tilde{x}}_{ij}},b_{\tilde{\tilde{x}}_{ij}},c_{\tilde{\tilde{x}}_{ij}};\theta_{l,\tilde{\tilde{x}}_{ij}},\theta_{r,\tilde{\tilde{x}}_{ij}}\right), i=1,2,\cdots n;\ j=1,2,\cdots s$$

$$\tilde{\tilde{y}}_{ij}=\left(a_{\tilde{\tilde{y}}_{ij}},b_{\tilde{\tilde{y}}_{ij}},c_{\tilde{\tilde{y}}_{ij}};\theta_{l,\tilde{\tilde{y}}_{ij}},\theta_{r,\tilde{\tilde{y}}_{ij}}\right),\ i=1,2,\cdots,n;\ j=1,2,\cdots,s$$

The problem with type – 2 fuzzy input and output need to be reduced by any type reduction method.

A. Using Infimum of Upper Reduction Method

Case-i : Theorem 1 can be used to defuzzyfy the T2DEA model and corresponding crisp formulation using upper reduction method takes the form when $\beta\in\left[0,\frac{1+(1-\alpha)\theta_r}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0\\
Subject\ to\\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{k0}}+\frac{2\beta\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_r}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(a_{\tilde{x}_{io}}+\frac{2\beta\left(b_{\tilde{x}_{io}}-a_{\tilde{x}_{io}}\right)}{1+(1-\alpha)\theta_r}\right)u_i\geq 0\\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{kj}}+\frac{2\beta\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_r}\right)v_k-\sum_{i=1}^{m}\left(a_{\tilde{x}_{ij}}+\frac{2\beta\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_r}\right)u_i\geq 0\\
u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
j=1,2,....,n
\end{array}\right\}$$

Case-ii : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1+(1-\alpha)\theta_r}{4},\frac{1}{2}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0\\
Subject\ to\\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}-\frac{(1-2\beta)\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}}-\frac{(1-2\beta)\left(b_{\tilde{x}_{i0}}-a_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\geq 0\\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}-\frac{(1-2\beta)\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}-\frac{(1-2\beta)\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\geq 0\\
u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
j=1,2,....,n
\end{array}\right\}$$

Case-iii : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1}{2},\frac{3-(1-\alpha)\theta_r}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0\\
Subject\ to\\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\geq 0\\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\leq 0\\
u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
j=1,2,....,n
\end{array}\right\}$$

Case-iv **:** Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{3-(1-\alpha)\theta_r}{4},1\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0\\
Subject\ to\\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{k0}}-\frac{2(1-\beta)\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_r}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(c_{\tilde{x}_{i0}}-\frac{2(1-\beta)\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_r}\right)u_i\geq 0\\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{kj}}-\frac{2(1-\beta)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_r}\right)v_k-\sum_{i=1}^{m}\left(c_{\tilde{x}_{ij}}-\frac{2(1-\beta)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_r}\right)u_i\geq 0\\
u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
j=1,2,....,n
\end{array}\right\}$$

B. Using Supremum of Upper Reduction Method

Case-i : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta \in \left[0, \frac{1+(1-\alpha)\theta_r}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{k0}} - \frac{2\beta\left(c_{\tilde{y}_{k0}} - b_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_r}\right)v_k - \tilde{f}\sum_{i=1}^{m}\left(c_{\tilde{x}_{i0}} - \frac{2\beta\left(c_{\tilde{x}_{i0}} - b_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_r}\right)u_i \geq 0 \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{kj}} - \frac{2\beta\left(c_{\tilde{y}_{kj}} - b_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_r}\right)v_k - \sum_{i=1}^{m}\left(c_{\tilde{x}_{ij}} - \frac{2\beta\left(c_{\tilde{x}_{ij}} - b_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_r}\right)u_i \geq 0 \\
u_1, u_2, ..., u_m \geq 0, v_1, v_2, ..., v_s \geq 0 \\
j = 1, 2, ..., n
\end{array}\right\}$$

Case-ii : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta \in \left[\frac{1+(1-\alpha)\theta_r}{4}, \frac{1}{2}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}} + \frac{(1-2\beta)\left(c_{\tilde{y}_{k0}} - b_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_r}\right)v_k - \tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}} + \frac{(1-2\beta)\left(c_{\tilde{x}_{i0}} - b_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_r}\right)u_i \geq 0 \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}} + \frac{(1-2\beta)\left(c_{\tilde{y}_{kj}} - b_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_r}\right)v_k - \sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}} + \frac{(1-2\beta)\left(c_{\tilde{x}_{ij}} - b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i \geq 0 \\
u_1, u_2, ..., u_m \geq 0, v_1, v_2, ..., v_s \geq 0 \\
j = 1, 2, ..., n
\end{array}\right\}$$

Case-iii : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta \in \left[\frac{1}{2}, \frac{3-(1-\alpha)\theta_r}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}} - \frac{(2\beta-1)\left(b_{\tilde{y}_{k0}} - a_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_r}\right)v_k - \tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}} - \frac{(2\beta-1)\left(b_{\tilde{x}_{i0}} - a_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_r}\right)u_i \geq 0 \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}} - \frac{(2\beta-1)\left(b_{\tilde{y}_{kj}} - a_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_r}\right)v_r - \sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}} - \frac{(2\beta-1)\left(b_{\tilde{x}_{ij}} - a_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i \leq 0 \\
u_1, u_2, ..., u_m \geq 0, v_1, v_2, ..., v_s \geq 0 \\
j = 1, 2, ..., n
\end{array}\right\}$$

Case-iv : Theorem 3 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta \in \left[\frac{3-(1-\alpha)\theta_r}{4}, 1\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{k0}} + \frac{2(1-\beta)\left(b_{\tilde{y}_{k0}} - a_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_r}\right)v_k - \tilde{f}\sum_{i=1}^{m}\left(a_{\tilde{x}_{i0}} + \frac{2(1-\beta)\left(b_{\tilde{x}_{i0}} - a_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_r}\right)u_i \geq 0 \\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{kj}} + \frac{2(1-\beta)\left(b_{\tilde{y}_{kj}} - a_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_r}\right)v_k - \sum_{i=1}^{m}\left(a_{\tilde{x}_{ij}} + \frac{2(1-\beta)\left(b_{\tilde{x}_{ij}} - a_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_r}\right)u_i \geq 0 \\
u_1, u_2, ..., u_m \geq 0, v_1, v_2, ..., v_s \geq 0 \\
j = 1, 2, ..., n
\end{array}\right\}$$

C. Using Infimum of Lower Reduction Method

Case-i : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta \in \left[0, \frac{1-(1-\alpha)\theta_l}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{k0}}+\frac{2\beta\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(a_{\tilde{x}_{i0}}+\frac{2\beta\left(b_{\tilde{x}_{i0}}-a_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
\sum_{k=1}^{s}\left(a_{\tilde{y}_{kj}}+\frac{2\beta\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\sum_{i=1}^{m}\left(a_{\tilde{x}_{ij}}+\frac{2\beta\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
u_1,u_2,....,u_m\geq 0, v_1,v_2,....,v_s\geq 0 \\
j=1,2,....,n
\end{array}\right\}$$

Case-ii : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1-(1-\alpha)\theta_l}{4},\frac{1}{2}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}-\frac{(1-2\beta)\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}}-\frac{(1-2\beta)\left(b_{\tilde{x}_{i0}}-a_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0 \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}-\frac{(1-2\beta)\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}-\frac{(1-2\beta)\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0 \\
u_1,u_2,....,u_m\geq 0, v_1,v_2,....,v_s\geq 0 \\
j=1,2,....,n
\end{array}\right\}$$

Case-iii : Theorem 4 can be used to defuzzyfy the DEA model (34) and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1}{2},\frac{3+(1-\alpha)\theta_l}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0 \\
\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}-\frac{(2\beta-1)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0 \\
u_1,u_2,....,u_m\geq 0, v_1,v_2,....,v_s\geq 0 \\
j=1,2,....,n
\end{array}\right\}$$

Case-iv : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{3+(1-\alpha)\theta_l}{4},1\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{k0}}-\frac{2(1-\beta)\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(c_{\tilde{x}_{i0}}-\frac{2(1-\beta)\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{kj}}-\frac{2(1-\beta)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\sum_{i=1}^{m}\left(c_{\tilde{x}_{ij}}-\frac{2(1-\beta)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
u_1,u_2,....,u_m\geq 0, v_1,v_2,....,v_s\geq 0 \\
j=1,2,....,n
\end{array}\right\}$$

D. Using Supremum of Lower Reduction Method

Case-i : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[0,\frac{1-(1-\alpha)\theta_l}{4}\right]$

$$\left.\begin{array}{l}
Max\ \tilde{f}_0 \\
Subject\ to \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{k0}}-\frac{2\beta\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(c_{\tilde{x}_{i0}}-\frac{2\beta\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
\sum_{k=1}^{s}\left(c_{\tilde{y}_{kj}}-\frac{2\beta\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_l}\right)v_r-\sum_{i=1}^{m}\left(c_{\tilde{x}_{ij}}-\frac{2\beta\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0 \\
u_1,u_2,....,u_m\geq 0, v_1,v_2,....,v_s\geq 0 \\
j=1,2,....,n
\end{array}\right\}$$

Case-ii : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1-(1-\alpha)\theta_l}{4},\frac{1}{2}\right]$

$$\left.\begin{aligned}
&Max\ \tilde{f}_0\\
&Subject\ to\\
&\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}+\frac{(1-2\beta)\left(c_{\tilde{y}_{k0}}-b_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{i0}}+\frac{(1-2\beta)\left(c_{\tilde{x}_{i0}}-b_{\tilde{x}_{i0}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0\\
&\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}+\frac{(1-2\beta)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_l}\right)v_r-\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}+\frac{(1-2\beta)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0\\
&u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
&j=1,2,....,n
\end{aligned}\right\}$$

Case-iii : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{1}{2},\frac{3+(1-\alpha)\theta_l}{4}\right]$

$$\left.\begin{aligned}
&Max\ \tilde{f}_0\\
&Subject\ to\\
&\sum_{k=1}^{s}\left(b_{\tilde{y}_{k0}}-\frac{(2\beta-1)\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{k0}}-\frac{(2\beta-1)\left(b_{\tilde{x}_{k0}}-a_{\tilde{x}_{k0}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0\\
&\sum_{k=1}^{s}\left(b_{\tilde{y}_{kj}}-\frac{(2\beta-1)\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1+(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(b_{\tilde{x}_{ij}}-\frac{(2\beta-1)\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1+(1-\alpha)\theta_l}\right)u_i\geq 0\\
&u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
&j=1,2,....,n
\end{aligned}\right\}$$

Case-iv : Theorem 4 can be used to defuzzyfy the DEA model and the corresponding crisp formulation takes the form when $\beta\in\left[\frac{3+(1-\alpha)\theta_l}{4},1\right]$

$$\left.\begin{aligned}
&Max\ \tilde{f}_0\\
&Subject\ to\\
&\sum_{k=1}^{s}\left(a_{\tilde{y}_{k0}}+\frac{2(1-\beta)\left(b_{\tilde{y}_{k0}}-a_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\tilde{f}\sum_{i=1}^{m}\left(a_{\tilde{x}_{i0}}+\frac{2(1-\beta)\left(b_{\tilde{x}_{i0}}-a_{\tilde{x}_{i0}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0\\
&\sum_{k=1}^{s}\left(a_{\tilde{y}_{rj}}+\frac{2(1-\beta)\left(b_{\tilde{y}_{kj}}-a_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_l}\right)v_k-\sum_{i=1}^{m}\left(a_{\tilde{x}_{ij}}+\frac{2(1-\beta)\left(b_{\tilde{x}_{ij}}-a_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_l}\right)u_i\geq 0\\
&u_1,u_2,....,u_m\geq 0,v_1,v_2,....,v_s\geq 0\\
&j=1,2,....,n
\end{aligned}\right\}$$

IV. NUMERICAL EXAMPLE

A numerical example is undertaken to a system a system consisting with five DMUs and each DMU with four similar inputs and four similar outputs. Then using the infimum of upper reduction method the proposed model can be written as:

$$\left.\begin{aligned}
&Max\ \tilde{f}_j\\
&\text{Subject to}\\
&\sum_{k=1}^{4}\left(b_{\tilde{y}_{kj}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{k0}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\tilde{f}_j\sum_{i=1}^{4}\left(b_{\tilde{x}_{ij}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\geq 0\\
&\lambda_j\left(u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4\right)\leq 0,where\\
&\lambda_j=\sum_{k=1}^{4}\left(b_{\tilde{y}_{kj}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{kj}}-b_{\tilde{y}_{kj}}\right)}{1-(1-\alpha)\theta_r}\right)v_k-\sum_{i=1}^{4}\left(b_{\tilde{x}_{ij}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{ij}}-b_{\tilde{x}_{ij}}\right)}{1-(1-\alpha)\theta_r}\right)u_i\\
&u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4\geq 0\\
&j=1,2,....,n
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\lambda_1=\ &\left(b_{\tilde{y}_{11}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{11}}-b_{\tilde{y}_{11}}\right)}{1-(1-\alpha)\theta_r}\right)v_1+\left(b_{\tilde{y}_{21}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{21}}-b_{\tilde{y}_{21}}\right)}{1-(1-\alpha)\theta_r}\right)v_2+\\
&\left(b_{\tilde{y}_{31}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{31}}-b_{\tilde{y}_{31}}\right)}{1-(1-\alpha)\theta_r}\right)v_3+\left(b_{\tilde{y}_{41}}+\frac{(2\beta-1)\left(c_{\tilde{y}_{41}}-b_{\tilde{y}_{41}}\right)}{1-(1-\alpha)\theta_r}\right)v_4-\\
&\left(b_{\tilde{x}_{11}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{11}}-b_{\tilde{x}_{11}}\right)}{1-(1-\alpha)\theta_r}\right)u_1-\left(b_{\tilde{x}_{21}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{21}}-b_{\tilde{x}_{21}}\right)}{1-(1-\alpha)\theta_r}\right)u_2-\\
&\left(b_{\tilde{x}_{31}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{31}}-b_{\tilde{x}_{31}}\right)}{1-(1-\alpha)\theta_r}\right)u_3-\left(b_{\tilde{x}_{41}}+\frac{(2\beta-1)\left(c_{\tilde{x}_{41}}-b_{\tilde{x}_{41}}\right)}{1-(1-\alpha)\theta_r}\right)u_4
\end{aligned}\right\}$$

$$\lambda_2 = \left.\begin{aligned}&\left(b_{\tilde{y}_{12}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{12}} - b_{\tilde{y}_{12}}\right)}{1-(1-\alpha)\theta_r}\right)v_1 + \left(b_{\tilde{y}_{22}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{22}} - b_{\tilde{y}_{22}}\right)}{1-(1-\alpha)\theta_r}\right)v_2 + \\ &\left(b_{\tilde{y}_{32}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{32}} - b_{\tilde{y}_{32}}\right)}{1-(1-\alpha)\theta_r}\right)v_3 + \left(b_{\tilde{y}_{42}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{42}} - b_{\tilde{y}_{42}}\right)}{1-(1-\alpha)\theta_r}\right)v_4 - \\ &\left(b_{\tilde{x}_{12}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{12}} - b_{\tilde{x}_{12}}\right)}{1-(1-\alpha)\theta_r}\right)u_1 - \left(b_{\tilde{x}_{22}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{22}} - b_{\tilde{x}_{22}}\right)}{1-(1-\alpha)\theta_r}\right)u_2 - \\ &\left(b_{\tilde{x}_{32}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{32}} - b_{\tilde{x}_{32}}\right)}{1-(1-\alpha)\theta_r}\right)u_3 - \left(b_{\tilde{x}_{42}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{41}} - b_{\tilde{x}_{42}}\right)}{1-(1-\alpha)\theta_r}\right)u_4\end{aligned}\right\}$$

$$\lambda_3 = \left.\begin{aligned}&\left(b_{\tilde{y}_{13}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{13}} - b_{\tilde{y}_{13}}\right)}{1-(1-\alpha)\theta_r}\right)v_1 + \left(b_{\tilde{y}_{23}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{23}} - b_{\tilde{y}_{23}}\right)}{1-(1-\alpha)\theta_r}\right)v_2 + \\ &\left(b_{\tilde{y}_{33}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{33}} - b_{\tilde{y}_{33}}\right)}{1-(1-\alpha)\theta_r}\right)v_3 + \left(b_{\tilde{y}_{43}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{43}} - b_{\tilde{y}_{43}}\right)}{1-(1-\alpha)\theta_r}\right)v_4 - \\ &\left(b_{\tilde{x}_{13}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{13}} - b_{\tilde{x}_{13}}\right)}{1-(1-\alpha)\theta_r}\right)u_1 - \left(b_{\tilde{x}_{23}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{23}} - b_{\tilde{x}_{23}}\right)}{1-(1-\alpha)\theta_r}\right)u_2 - \\ &\left(b_{\tilde{x}_{33}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{33}} - b_{\tilde{x}_{33}}\right)}{1-(1-\alpha)\theta_r}\right)u_3 - \left(b_{\tilde{x}_{43}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{43}} - b_{\tilde{x}_{43}}\right)}{1-(1-\alpha)\theta_r}\right)u_4\end{aligned}\right\}$$

$$\lambda_4 = \left.\begin{aligned}&\left(b_{\tilde{y}_{14}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{14}} - b_{\tilde{y}_{14}}\right)}{1-(1-\alpha)\theta_r}\right)v_1 + \left(b_{\tilde{y}_{24}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{24}} - b_{\tilde{y}_{24}}\right)}{1-(1-\alpha)\theta_r}\right)v_2 + \\ &\left(b_{\tilde{y}_{34}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{34}} - b_{\tilde{y}_{34}}\right)}{1-(1-\alpha)\theta_r}\right)v_3 + \left(b_{\tilde{y}_{44}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{44}} - b_{\tilde{y}_{44}}\right)}{1-(1-\alpha)\theta_r}\right)v_4 - \\ &\left(b_{\tilde{x}_{14}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{14}} - b_{\tilde{x}_{14}}\right)}{1-(1-\alpha)\theta_r}\right)u_1 - \left(b_{\tilde{x}_{24}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{24}} - b_{\tilde{x}_{24}}\right)}{1-(1-\alpha)\theta_r}\right)u_2 - \\ &\left(b_{\tilde{x}_{34}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{34}} - b_{\tilde{x}_{34}}\right)}{1-(1-\alpha)\theta_r}\right)u_3 - \left(b_{\tilde{x}_{44}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{44}} - b_{\tilde{x}_{44}}\right)}{1-(1-\alpha)\theta_r}\right)u_4\end{aligned}\right\}$$

$$\lambda_5 = \left.\begin{aligned}&\left(b_{\tilde{y}_{15}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{15}} - b_{\tilde{y}_{15}}\right)}{1-(1-\alpha)\theta_r}\right)v_1 + \left(b_{\tilde{y}_{25}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{25}} - b_{\tilde{y}_{25}}\right)}{1-(1-\alpha)\theta_r}\right)v_2 + \\ &\left(b_{\tilde{y}_{35}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{35}} - b_{\tilde{y}_{35}}\right)}{1-(1-\alpha)\theta_r}\right)v_3 + \left(b_{\tilde{y}_{45}} + \frac{(2\beta-1)\left(c_{\tilde{y}_{45}} - b_{\tilde{y}_{45}}\right)}{1-(1-\alpha)\theta_r}\right)v_4 - \\ &\left(b_{\tilde{x}_{15}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{15}} - b_{\tilde{x}_{15}}\right)}{1-(1-\alpha)\theta_r}\right)u_1 - \left(b_{\tilde{x}_{25}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{25}} - b_{\tilde{x}_{25}}\right)}{1-(1-\alpha)\theta_r}\right)u_2 - \\ &\left(b_{\tilde{x}_{35}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{35}} - b_{\tilde{x}_{35}}\right)}{1-(1-\alpha)\theta_r}\right)u_3 - \left(b_{\tilde{x}_{45}} + \frac{(2\beta-1)\left(c_{\tilde{x}_{45}} - b_{\tilde{x}_{45}}\right)}{1-(1-\alpha)\theta_r}\right)u_4\end{aligned}\right\}$$

Table II: List of Decision Making Units under consideration

DMU Number	Name of Indian Mutual Fund ELSS Tax savings Scheme
DMU1	Reliance Tax Saver Fund-Growth
DMU2	Axis Long Term Equity Fund-Growth
DMU3	ICICI Prudential Rewards of Investing and Generation of Healthy Tax Savings Fund-Growth
DMU4	ICICI Prudential Tax Plan Regular-Growth
DMU5	Franklin India Tax shield-Growth

Table III: List of Inputs

Input Number	Input Type
Input 1	Asset under Management (AUM)
Input 2	Expense Ratio
Input 3	1 Yr CNX Nifty Benchmark Return
Input 4	1 Yr Fund Return

Table IV: List of Outputs

Output Number	Output Type
Output 1	Mean Return
Output 2	Standard Deviation
Output 3	Sharpe Ratio
Output 4	Information Ratio

Table V: Input data for five DMU

DMUi	Input1	Input2	Input3	Input4
DMU1	$(20.0,25.0,30.0,\theta_l,\theta_r)$	$(2.00,2.35,3.10,\theta_l,\theta_r)$	$(28.00,28.50,29.00,\theta_l,\theta_r)$	$(90.00,91.50,93.00,\theta_l,\theta_r)$
DMU2	$(15.0,20.0,25.0,\theta_l,\theta_r)$	$(2.64.0,2.74,2.84,\theta_l,\theta_r)$	$(28.00,28.50,29.00,\theta_l,\theta_r)$	$(60.00,61.50,63.00,\theta_l,\theta_r)$
DMU3	$(0.5,0.6,0.7,\theta_l,\theta_r)$	$(2.50,2.70,3.10,\theta_l,\theta_r)$	$(28.00,28.50,29.00,\theta_l,\theta_r)$	$(55.00,60.00,65.00,\theta_l,\theta_r)$
DMU4	$(1.5,1.6,1.7,\theta_l,\theta_r)$	$(2.36,2.46,2.56,\theta_l,\theta_r)$	$(28.00,28.50,29.00,\theta_l,\theta_r)$	$(50.00,55.00,60.00,\theta_l,\theta_r)$
DMU5	$(12.5,13.5,16.5,\theta_l,\theta_r)$	$(2.00,2.50,3.00,\theta_l,\theta_r)$	$(28.00,28.50,29.00,\theta_l,\theta_r)$	$(45.00,47.00,49.00,\theta_l,\theta_r)$

Table VI: Output data for five DMU

DMUi	Output 1	Output 2	Output 3	Output 4
DMU1	$(28.0,29.0,30.0,\theta_l,\theta_r)$	$(22.00,24.00,26.00,\theta_l,\theta_r)$	$(0.80,0.85,0.95,\theta_l,\theta_r)$	$(0.84,0.86,0.88,\theta_l,\theta_r)$
DMU2	$(25.0,27.0,28.0,\theta_l,\theta_r)$	$(12.00,14.00,16.00,\theta_l,\theta_r)$	$(1.20,1.50,1.70,\theta_l,\theta_r)$	$(0.56,0.58,0.60,\theta_l,\theta_r)$
DMU3	$(25.0,27.0,28.0,\theta_l,\theta_r)$	$(12.50,14.50,16.50,\theta_l,\theta_r)$	$(1.30,1.35,1.40,\theta_l,\theta_r)$	$(1.15,1.20,1.25,\theta_l,\theta_r)$
DMU4	$(20.0,23.0,26.0,\theta_l,\theta_r)$	$(17.50,18.50,19.50,\theta_l,\theta_r)$	$(0.95,0.97,1.00,\theta_l,\theta_r)$	$(0.95,0.97,0.99,\theta_l,\theta_r)$
DMU5	$(18.0,20.0,22.0,\theta_l,\theta_r)$	$(13.50,14.50,15.50,\theta_l,\theta_r)$	$(0.92,0.94,0.96,\theta_l,\theta_r)$	$(0.92,0.94,0.96,\theta_l,\theta_r)$

Table VII: Evaluation of Result

DMU i	Efficiency Score	Rank
DMU 1	0.7750428	4
DMU 2	0.8070967	2
DMU 3	0.7834569	3
DMU 4	0.8096789	1
DMU 5	0.7634527	5

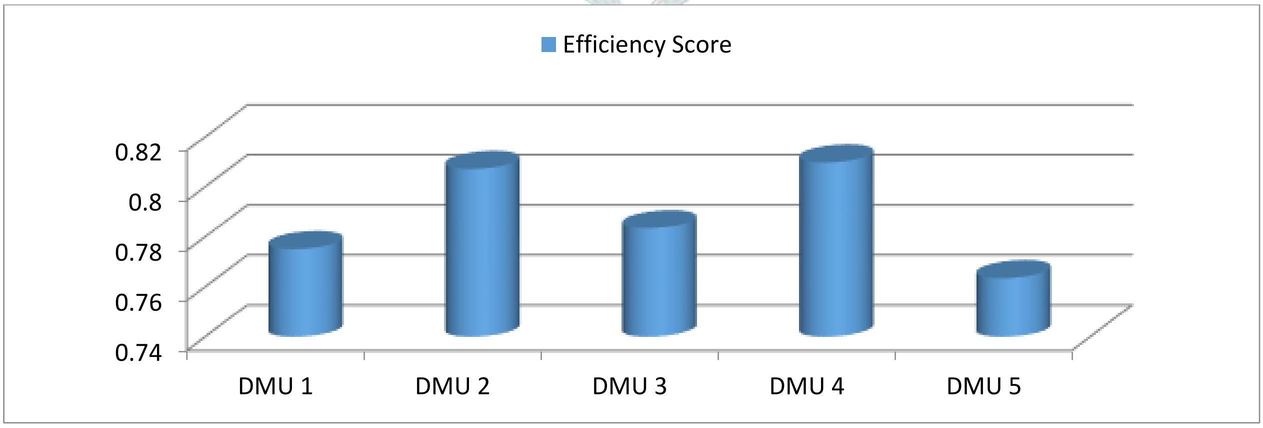

Fig.2 Efficiency scores of DMUs

V. RESULT AND DISCUSSION

In this paper we have ranked five mutual funds from India selected on tax savings funds (ELSS-funds). Considering uncertainty in the inputs and outputs, the data are taken as type-2 fuzzy triangular set to add extra dimension than type-1 fuzzy set. The data used for inputs and outputs in the problem can also be taken as trapezoidal type-2 fuzzy set or normal type2 fuzzy set and accordingly the problem can also be developed. The type reduction we have adopted VaR reduction method. Similar result can be obtained if we use CV reduction method. We belief that the method proposed in the paper gives a comprehensive and efficient result, so that investor can take their decision confidently to invest to these type of funds. We can also apply the present model to any number of funds by making similar extension of the problem. The present model can also be extended to different types of decision making problems including ranking mutual funds, performance of equity stocks or project selection.

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