METRIC INDUCED TOPOLOGICAL SPACE OF COMPLEX FUZZY MATRICES BY THE USE OF FROBENIUS INNER PRODUCT

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Abstract: **In this paper we extend the definitions of Frobenius inner product, Frobenius Norm and the induced metric of usual matrices to Frobenius inner product, Frobenius Norm and the induced metric definitions of complex fuzzy matrices. By using these definitions, we define Inner product space, Normed space and metric space of complex fuzzy matrices. Finally, in this paper we use some basic result from general topology and prove that the collection of all complex fuzzy square matrices will also a topological space.**

IndexTerms - **Complex Fuzzy Sets; Complex Fuzzy Matrices; Frobenius Inner Product; Frobenius Norm; Induced Metric; Topology.**

I. INTRODUCTION

The concept of fuzzy set theory was introduced by Zadeh [1] in 1965. Fuzzy set theory has many applications in various fields, especially in medicine and treatments. Fuzzy matrices are another important concept in fuzzy mathematics, which has many applications in modelling different uncertain situations in medical diagnosis and science.

The notion of fuzzy matrices was introduced by Thomson in 1977, and it is developed by Kim and Roush as an extension of Boolean matrices. The concept of complex fuzzy sets, which is an extension of the classical fuzzy sets, in which the membership function assigns its values on the unit disc of the complex plane [set of all complex numbers with modulus less than or equal to 1]. The representation of complex degree of membership in polar co-ordinates was introduced by Ramot. Frobenius inner product is a binary operation that gives a number in return when we give two matrices to it.

II. PRELIIMINARIES

2.1 Fuzzy Matrices

A fuzzy matrix is a matrix which has its elements from [0,1], called the fuzzy unit interval.

A fuzzy matrix A of order $m \times n$ is defined as $A = \left[\langle a_{ij}, a_{ij\mu}\rangle\right]_{m\times n}$, where $a_{ij\mu}$ is the membership value of the element a_{ij} in A.

For simplicity, we write A as $A = \left[a_{ij}\right]_{m\times n}$.

2.2 Complex Fuzzy Set

A complex fuzzy set defined on the universe of discourse U, is characterized by a membership function $\mu_F(x)$, that assign any element $a \in F$ a complex valued grade of membership in F. By definition, the values $\mu_F(x)$ receive all lie within the unit circle in the complex plane, and are thus of the form $r_F(x)e^{i\omega_F(x)}, i = \sqrt{-1}$, $r_F(x)$ and $\omega_F(x)$ are both real valued $r_F(x) \epsilon$ [0, 1] , $\omega_F(x) \epsilon$ [0, 2π) , the complex fuzzy set may be represented as the set of ordered pairs. F = {(x,$\mu_F(x)$)): xϵ U}

2.3 Complex Fuzzy Matrices

A complex fuzzy matrix is a matrix which has its elements from the unit disc in the complex plane. A complex fuzzy matrix of order m × n is defined as, $CF = \left[\langle a_{ij}, a_{ij\mu}\rangle\right]_{m\times n}$ $a_{ij\mu} = r_{ij}e^{i\omega_{ij}}$; $r_{ij}\epsilon\,[0,1]$ and $\omega_{ij}\epsilon\,\epsilon[0,2\pi)$

$$\text{Example, } CF = \begin{bmatrix} 0.1e^{i\frac{\pi}{2}} & 0.5e^{i\pi} & 0 \\ 0.3e^{i\frac{\pi}{3}} & 0.2e^{i0} & 0.7e^{i\frac{\pi}{4}} \\ 0e^{i0} & 0.6e^{i\pi} & 1 \end{bmatrix}_{3\times 3}$$

2.4) OPERATIONS ON THE COMPLEX FUZZY MATRICES

2.4.1 Addition of two complex fuzzy matrices

Two complex fuzzy matrices are comfortable for addition if the matrices are of same order. If $CF_1 = [a_{ij\mu}]_{m\times n}$ and $CF_2 = [b_{ij\vartheta}]_{m\times n}$ then $CF_1 + CF_2 = [c_{ij\eta}]_{m\times n}$; $c_{ij\eta} = max\{a_{ij\mu}, b_{ij\vartheta}\}$

2.4.2 Multiplication of two complex fuzzy matrices

The product of two complex fuzzy matrices is to be defined we need the number of columns of the first matrix is equal to the number of rows of the second matrix. i.e.; If $CF_1 = [a_{ij\mu}]_{m\times n}$ and $CF_2 = [b_{ij\vartheta}]_{n\times p}$; $CF = (CF_1)(CF_2) = [c_{ij\gamma}]_{m\times p}$;

$$c_{ij\gamma} = max\{min\{a_{i1\mu}, b_{1j\vartheta}\}, min\{a_{i2\mu}, b_{2j\vartheta}\}, min\{a_{i3\mu}, b_{3j\vartheta}\}, \ldots, min\{a_{in\mu}, b_{1n\vartheta}\}\}; 1 \le i \le m, 1 \le j \le p.$$

2.4.3 Trace of a complex fuzzy matrix

Let $CF = [a_{ij\mu}]_{n\times n}$ be a square complex fuzzy matrix. Then $Trace(CF) = max\{a_{ii\mu}\}$

2.4.4 Conjugate transpose of a complex fuzzy matrix

The conjugate transpose of a complex fuzzy matrix CF is defined as the transpose of the conjugate of the matrix.

That is $\overline{CF}^T = (CF)^*$.

2.4.5 Determinant of a complex fuzzy matrix

Let $CF = [a_{ij\mu}]_{n\times n}$ be an n × n complex fuzzy matrix then, $||CF| = a_{11\mu}M_{11} + a_{12\mu}M_{12}+. \; . \; .+a_{1n\mu}M_{1n}$, where M_{ij} is the minor of the $(ij)^{th}$ entry.

III. THEOREMS ON THE COLLECTION OF COMPLEX FUZZY MATRICES

3.1 Theorem I: -Vector space of complex fuzzy matrices

The set of all complex fuzzy matrices of order $n \times n$ will form a vector space over the field, the unit disc of the complex plane under the binary operations,

(i) Addition of two complex fuzzy matrices: -

If $CF_1 = [a_{ij\mu}]_{m\times n}$ and $CF_2 = [b_{ij\vartheta}]_{m\times n}$ then $CF_1 + CF_2 = [c_{ij\eta}]_{m\times n}$; $c_{ij\eta} = max\{a_{ij\mu}, b_{ij\vartheta}\}$

(ii) Scalar multiplication: -

If $CF = [a_{ij\mu}]_{n\times n}$, then for any scalar $z = re^{i\theta}$ in the unit disc of the complex plane,

The scalar multiplication is defined by,

$z.CF = [c_{ij\gamma}]_{n\times n}; \; c_{ij\gamma} = min\{re^{i\theta}, r_{ij}e^{i\omega_{ij}}\}$

Proof: -

Closure Property

By the definition itself

Associativity

Since the minimum operator is associative.

Existence of additive identity

The zero matrix is the additive identity.

Existence of additive inverse

If $CF = [a_{ij\mu}]_{n\times n}$, the additive identity of CF is $-CF$.

i.e.; If $CF_1 = [a_{ij\mu}]_{n\times n}$ and $CF_2 = [b_{ij\vartheta}]_{n\times n}$; $a_{ij\mu} = a_{ij}e^{i\omega_{ij}}$ and $b_{ij\mu} = b_{ij}e^{i\vartheta_{ij}}$

$CF_1 - CF_2 = [c_{ij\gamma}]_{n\times n} \dashrightarrow (\#)$

$$c_{ij\gamma} = \begin{cases} a_{ij}e^{i\omega_{ij}} & ; \; a_{ij} > b_{ij} \; and \; \omega_{ij} > \vartheta_{ij} \\ 0e^{i\omega_{ij}} & ; \; a_{ij} \le b_{ij} \; and \; \omega_{ij} > \vartheta_{ij} \\ 0e^{i0} & ; \; a_{ij} \le b_{ij} \; and \; \omega_{ij} \le \vartheta_{ij} \\ a_{ij}e^{i0} & ; \; a_{ij} > b_{ij} \; and \; \omega_{ij} > \vartheta_{ij} \end{cases}$$

So, if we take CF^{-1} as $-CF$,

$CF - CF = [c_{ij\gamma}]_{n\times n}$; $c_{ij\gamma} = 0e^{i0}$, $\forall i, j$

Since, $a_{ij} = b_{ij} \; and \; \omega_{ij} = \vartheta_{ij}$

So, the collection of all complex fuzzy matrices of order $n \times n$ is a group under addition.

Closure property of scalar multiplication

By the definition itself.

Right and left distributive properties holds

$z.(CF_1 + CF_2) = z.CF_1 + z.CF_2$

$(CF_1 + CF_2).z = CF_1.z + CF_2.z$

Since the minimum operator id distributive over maximum operator.

Associativity holds

$(z_1z_2).CF = z_1.(z_2.CF)$

Since the minimum operator is associative.

Scalar multiplication by unity

$1e^{i0}.a_{ij\mu} = \min\{1e^{i0}, r_{ij}e^{i\omega_{ij}}\} = r_{ij}e^{i\omega_{ij}}$

i.e.; $1e^{i0}.A = A$.

$CF^{n\times n}$ is a vector space under the defined operations.

Hence the theorem.

3.2 Frobenius inner product of complex fuzzy matrices

The Frobenius inner product of complex fuzzy matrices is defined by,

for any two complex fuzzy matrices CF_1 and CF_2 of order $n \times n$ then,

$\langle CF_1, CF_2\rangle = Trace((CF_1)^*(CF_2))$, where $(CF_1)^*$ is the conjugate transpose of CF_1.

Theorem II: - The inner product space of complex fuzzy matrices.

The collection of all complex fuzzy matrices of order $n \times n$, $CF^{n\times n}$ will form an inner product space under the mapping

$\langle CF_1, CF_2\rangle = Trace((CF_1)^*(CF_2)) \to (1)$

For any $CF_1, CF_2 \in CF^{n\times n}$ and where $(CF_1)^*$ is the conjugate transpose of CF_1.

Proof: -

Using Theorem: - I, we have $CF^{n\times n}$ is a vector space over the field D (unit disc in the complex plane.)

It remains to show that the mapping defined in (1) is an inner product.

$\langle \, , \rangle : \; CF^{n\times n} \times CF^{n\times n} \longrightarrow D$ where D is the unit disc in the complex plane.

Closure property

Trace is a scalar in the unit disc, therefore, $\langle CF_1, CF_2\rangle \in D$

Linearity

Since the trace is a linear function, therefore $\langle \, , \rangle$ is linear.

Conjugate symmetry

$\langle CF_1, CF_2\rangle = \overline{\langle CF_2, CF_1\rangle}$

$\overline{\langle CF_2, CF_1\rangle} = \overline{(Trace((CF_2)^*(CF_1))}$

$= (Trace((CF_1)^*(CF_2))$

$= \langle CF_1, CF_2\rangle$

Positive definiteness

$\langle CF, CF\rangle > 0 \quad , \forall\, CF \neq O$

Since the Trace is defined as $\sum a_{ij\mu} = Max\{\, a_{ij\mu}\}$

Therefore, $CF^{n\times n}$ is an inner product space.

Hence the theorem.

3.3 Frobenius norm on the complex fuzzy matrices

Frobenius norm on $CF^{n\times n}$ is a function $\|.\|$: $CF^{n\times n} \longrightarrow D$ defines by,

$\|CF\| = \sqrt{\langle CF, CF\rangle} = \sqrt{Trace((CF)^*(CF))} \longrightarrow (2)$, where CF^* is the conjugate transpose of CF.

Theorem III: - Normed space of complex fuzzy matrices

The collection $CF^{n\times n}$ of all complex fuzzy matrices of order $n \times n$ will form a normed space under the function (2) defined on $CF^{n\times n}$.

Proof: -

Using Theorem, I, $CF^{n\times n}$ is a vector space and also the function defined in (2) is well defined from $CF^{n\times n} \longrightarrow D$.

It remains to show that $\|.\|$ is a norm.

Sub Additivity

It is clear from the definition itself,

$\|CF_1 + CF_2\| = \sqrt{\langle CF_1 + CF_2, CF_1 + CF_2\,\rangle}$

$\leq \sqrt{\langle CF_1, CF_1\rangle} + \sqrt{\langle CF_2, CF_2\rangle}$

$= \|CF_1\| + \|CF_2\|$

Homogeneity

$\|z. CF\| = |Z|\,\|CF\|$, is clear from the definition itself.

Positive definiteness

Obvious from the definition itself.

Therefore, $CF^{n\times n}$ is a normed space.

Hence the theorem.

Theorem III: - Metric space of complex fuzzy matrices

The metric induced by the Frobenius nor will made the collection $CF^{n\times n}$ a metric space.

The induced metric is defined as,

$d: CF^{n\times n} \times CF^{n\times n} \longrightarrow D$ and

$d(CF_1, CF_2\,) = \|CF_1 - CF_2\|$, the '- 'operator already defined in (#).

Proof: -

Clearly d is a metric on $CF^{n\times n}$.

Positive definiteness

$d(CF_1, CF_2\,) \geq 0 \,, \forall\, CF_1, CF_2 \in\; CF^{n\times n}$, since the norm is.

Distinguishes points

$d(CF_1, CF_1\,) = 0 \; and \; d(CF_1, CF_2\,) \neq 0 \; where \; CF_1 \neq CF_2 \,, \forall\, CF_1, CF_2 \in\; CF^{n\times n}$.

Symmetry

Using the definition of norm.

Triangle inequality

Using the definition of norm.

Therefore, $CF^{n\times n}$ is metric space.

Hence the theorem'

3.4 Proposition *

Every metric space is topological space.

Proof: -

Let (X, d) be a metric space.

Let τ be the collection of all open sets in X.

Claim: - τ be a topology on X.

$\boldsymbol{\emptyset, X \in \tau}$

Since $\emptyset$ contains no points, we can say that $\emptyset$ contains all its open spheres centered on the elements of it. So $\emptyset \in \tau$.

Similarly, since X is the full set, it contains all the open spheres centered on the elements of X. So $X \in \tau$.

$\boldsymbol{\tau}$ is closed under arbitrary union

Let $\{U_\alpha : \alpha \in I\}$ be an arbitrary collection of open sets in τ.

If $\bigcup_{\alpha\in I} U_\alpha = \emptyset \Rightarrow \bigcup_{\alpha\in I} U_\alpha \in \tau$

So, let $\bigcup_{\alpha\in I} U_\alpha \neq \emptyset \Rightarrow let\; x \in \bigcup_{\alpha\in I} U_\alpha$

$\Rightarrow x \in U_\alpha$ for some $\alpha \in I$, open set in τ.

$\Rightarrow \exists\; x \in S_{r_\alpha}(x) \subseteq U_\alpha$.

$\Rightarrow x \in S_{r_\alpha}(x) \subseteq U_\alpha \subseteq \bigcup_{\alpha\in I} U_\alpha$.

Since x is arbitrary, $\bigcup_{\alpha\in I} U_\alpha$ is open $\Rightarrow \bigcup_{\alpha\in I} U_\alpha \in \tau$.

τ is closed under arbitrary union.

τ is closed under finite intersection

Let $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}\}$ be any finite collection of open sets in τ.

Claim: - $\bigcap_{i=1}^{n} U_{\alpha_i} \in \tau$

If $\bigcap_{i=1}^{n} U_{\alpha_i} = \emptyset \in \tau$, nothing to prove.

So, let $\bigcap_{i=1}^{n} U_{\alpha_i} \neq \emptyset \Rightarrow x \in \bigcap_{i=1}^{n} U_{\alpha_i}$

$\Rightarrow x \in U_{\alpha_i}, \forall\, i = 1,2,3, \ldots, n$

$\Rightarrow \exists\, S_{r_{\alpha_i}}(x)\ such\ that\ x \in S_{r_{\alpha_i}}(x) \subseteq U_{\alpha_i}, \forall\, i = 1,2,3, \ldots, n.$

$\Rightarrow x \in S_{r_{\alpha_i}}(x) \subseteq U_{\alpha_i} \subseteq \bigcap_{i=1}^{n} U_{\alpha_i}$.

Since $x \in \bigcap_{i=1}^{n} U_{\alpha_i}$ is arbitrary.

$\bigcap_{i=1}^{n} U_{\alpha_i} \in \tau$

τ is closed under finite intersection.

Hence (X, d) be a Topological space with topology τ.

Theorem IV: - Topological space of complex fuzzy matrices

The space $CF^{n\times n}$ is a topological space

Proof: -

Using theorem III, $CF^{n\times n}$ is metric space under the induced metric d.

So, using proposition *, every metric space is a topological space.

We can say that, $CF^{n\times n}$ is a topological space.

Hence the theorem.

IV. CONCLUSION

The work presented in this paper is the interesting frame work of complex fuzzy matrices and its collection. Also, we discussed the definitions of Frobenius inner product, Frobenius norm and the induced metric on the collection of complex fuzzy matrices, by using some already defined definitions in usual matrices. So, by using these definitions on the collection of all $n \times n$ complex fuzzy matrices, we proved that the collection will become an inner product space, a normed space and hence a metric space. Using the general result every metric space is a topological space, finally we concluded that the set all complex fuzzy matrices of order $n \times n$ will form a Topological space.

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