Infinite Conservation law of a damped forced Gardner equation via Bäcklund transformation

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Abstract: This study envisages on many infinitely conservation laws for the damped forced Gardner equation. Based on the use of binary bell polynomial, a bilinear form and a Bäcklund transformation for the said equation with certain condition are derived. Then from the bilinear Bäcklund transformation, the many infinitely conservation law for this equation is obtained through decomposing the binary Bell polynomial. Here conservation laws are all local and depend on the coefficients of terms of the said equation.
Keywords: Damped forced Gardner equation, bilinear Bäcklund transformation, Infinite conservation law.

1. Introduction

In physical system, conservation law is a measurable properties which do not change when the system rolls over time. Mathematically the study of conservation laws are helpful in studying of integrability of non-linear partial differential equation. It is helpful for studying the solution of non-linear partial differential equation. Neother's theorem (Bessel-Hagen, 1921;Bluman and Kumei, 1989; Boyer, 1967; Noether, 1918: Olver,1986) states that action symmetries can be used to find out local conservation laws for a differential equation with a variational principle.The infinite conservation laws and conserved quantities can be obtained by various methods like Bäcklund transformation (Wadatietal, 1975), from Lax pairs (Zhang and Chen, 2002), Eigen function method (Konnoetal, 1974), scattering method(Zakharov and Shabat, 1972), trace identities method(Tsuchida and Wadati, 1998), quasi-differential operators method by SATA (Kajiwara et al, 1990) etc.

Generally Gardner equation is formed by the KdV quadratic non-linearity and modified KdV cubic non-linearity. Different physical phenomena like the dusty-waves in plasma medium (Khateretal, 1999), internal sea waves in shear flows(Grimshawetal, 2001), negative ion wave in plasma(Watanabe, 1984), fluid dynamic(Kakutani and Yamasaki, 1978; Marchant and Smyth, 1990) can characterized by this model equation. The Gardner equation is as follows

$$H_t+AHH_x+BH^2H_x+CH_{xxx}=0 \quad (1)$$

Where *A*, *B*, *C* are all constants coefficient parameters.

It is known that particles collision makes a damping effect in any physical environment. For instance resonant energy transfer between particles and electrostatic waves in plasma medium causes for making damping. In some experimental work on space plasma it is noticed the affect of externally applied different type of damping on plasma wave(Senetal, 2015; Aslanov and Yudintsev, 2015). In addition in certain circumstances like water flowing crosses the bottom, wave by ship(Grimshaw, 1997; deSzoeke, 2004; Grimshawetal, 2010; Singh and Rao, 1999; Mowafy, 2008; Masood, 2010; Grimshaw et al, 2002; Li and Xiao, 2013) may produce external forces. In such circumstances we focus on the damped force Gardner equation which is as follows

$$H_t+AHH_x+BH^2H_x+CH_{xxx}+LH=\Delta(t), \qquad (2)$$

where *A*, *B*, *C* are any constant parameters, whereas, the damping and forcing coefficients are represented by *L* and $\Lambda(t)$ respectively.

2. Preliminary of binary Bell polynomial

In this a brief concept of Bell polynomial (Bell, 1834; Lambert et al, 1997; Roger and Schief, 2002; Fan, 2011) and corresponding useful notation has been discussed. Suppose $g = g(x_1, x_2, \cdots, x_m)$ is a continuously differentiable function up to infinite time. Then the multi-dimensional general Bell polynomials are defined by

$$Y_{n_1x_1 \ldots\ldots n_kx_k}(g) = \exp(-g)\partial_{x_1}^{n_1} \ldots \ldots \partial_{x_k}^{n_k}\exp(g) \qquad (3)$$

and the multi-dimensional binary Bell polynomial is defined as

$$Y_{n_1x_1 \ldots\ldots n_kx_k}(V,W)=\begin{cases} V_{n_1x_1 \ldots\ldots n_kx_k}, & n_1+n_2 \ldots +n_k \ \ is\ odd \\ W_{n_1x_1 \ldots\ldots n_kx_k}, & n_1+n_2 \ldots +n_k \ \ is\ even \end{cases} \qquad (4)$$

Few examples are

,

$$Y_x = V_x, \quad Y_{xx} = W_{xx} + V_x^2, \quad Y_{xy} = W_{xy} + V_xV_y, \quad Y_{xxx} = V_{xxx}+3V_xW_{xx} + V_x^3, \quad Y_{xxy} = V_{xxy}+ 2V_xW_{xy} + V_x^2V_y + W_{xx}V_y \qquad (5)$$

Theorem1. *The connection between binary Bell polynomial* $Y_{n_1x_1 \ldots n_kx_k}(V,W)$ *and the standard Hirota's bilinear operators(Hirota, 1971; Hirota, 1976; Hirota, 1980) can be given by*

$$Y_{n_1x_1 \ldots\ldots n_kx_k}\left(V = \ln\frac{F}{G}, W = \ln FG\right) = (FG)^{-1}D_{x_1}^{n_1} \cdots D_{x_k}^{n_k}FG \qquad (6)$$

where $n_1 + \ldots + n_k \geq 1$ and operators $D_{x_1}^{n_1}, \ldots, D_{x_k}^{n_k}$ which are Hirota's bilinear operators, are *given by*

$$D_{x_1}^{n_1} \ldots D_{x_k}^{n_k} FG = (\partial_{x_1} - \partial_{x'_1})^{n_1} \ldots \left(\partial_{x_k} - \partial_{x'_k}\right)^{n_k} F(x_1, \ldots, x_k) * G(x'_1, \ldots, x'_k) \quad (7)$$

3. Bilinear Bäcklund transformation for the damped force-Gardner equation

Using the transformation

$$H = a(t)Vx + m(t) \quad (8)$$

Eq. 8 is reduced to the equation

$$V_{xt} + \left(A + 2Bm(t)\right)a(t)V_x V_{xx} + Ba(t)^2 V_x^2 V_{xx} + \left(A + Bm(t)\right)m(t)V_{2x} + CV_{4x} = 0 \quad (9)$$

with the choice of $a(t)$=$s_0 e^{-\int Ldt}$ and $m(t)$=$e^{-Lt}\int e^{Lt}\Delta(t)dt$. Integrating Eq.9 once w. r. t. *x*, one can get

$$V_t + \frac{A + 2Bm(t))a(t)}{2} V_x^2 + \frac{Ba(t)^2}{3} V_x^3 + A + Bm(t))m(t)V_x + CV_{3x} = 0 \quad (10)$$

with the choice of integrating constant as 0. To express Eq.10 into the Y-polynomial form, we need to set a constraint

$$Y_{2x}(V, W) = 0 \quad (11)$$

Then we have

$$Y_t(V, W) + \left(A + Bm(t)\right)m(t)Y_x(V, W) + CY_{3x}(V, W) = 0 \quad (12)$$

with the condition $A + 2Bm(t) = 0$ and $Ba(t)^2 = -6C$. The Eq.11 and Eq.12 constitutes the bilinear form of the df Gardner equation, which can be expressed in form of Hirota's operators under the transformation $V = ln\frac{F}{G}, W = lnFG$ as

$$D_x^2 FG = 0$$

$$[D_t + \left(A + Bm(t)\right)m(t)D_x + CD_x^3]FG = 0 \quad (13)$$

Assuming that (V', W') and (V, W) are two different solutions of the Eq.11 and Eq.12, then we will consider the following:

$$E_1 = Y_{2x}(V', W') - Y_{2x}(V, W) = 0 \quad (14)$$

$$E_2 = Y_t(V', W') - Y_t(V, W) + p(t)(Y_x(V', W') - Y_x(V, W)) + C(Y_{3x}(V', W') - Y_{3x}(V, W)) = 0 \quad (15)$$

where $p(t) = (A + Bm(t))m(t)$.

Introduce new variables

$$V_1 = ln\frac{G'}{G}, \quad V_2 = ln\frac{F'}{F}, \quad V_3 = ln\frac{G'}{F}, \quad V_4 = ln\frac{F'}{G}$$

$$W_1 = lnG'G,\ W_2 = lnF'F, W_3 = lnG'F,\quad W_4 = lnF'G \qquad (16)$$

Decoupling Eq. 14 as

$$Y_x(V_4, W_4) = c_1 e^{V_1 - V_2}, \qquad Y_x(V_3, W_3) = c_2 e^{V_2 - V_1} \qquad (17)$$

Where c_1, c_2 are arbitrary constants and also decoupling Eq. 15 provides

$$Y_t(V_1,\ W_1) + p(t) Y_x(V_1,\ W_1) + 3c_1c_2 Y_x(V_1,\ W_1) + CY_{3x}(V_1,\ W_1) = 0 \qquad (18)$$

$$Y_t(V_2,\ W_2) + p(t) Y_x(V_2,\ W_2) + 3c_1c_2 Y_x(V_2,\ W_2) + CY_{3x}(V_2,\ W_2) = 0 \qquad (19)$$

The Eq.17, Eq.18 and Eq.19 constitute the bilinear Backlund's Transformation for the damped force Gardner equation. One can express these in Hirota's operators as

$$D_x F'G = c_1 FG \qquad (20)$$

$$D_x FG' = c_2 F'G \qquad (21)$$

$$[D_t + (p(t) + 3c_1c_2) D_x + CD_x^3] F'F = 0 \qquad (22)$$

$$[\mathrm{D_t} + (\mathrm{p(t)} + 3\mathrm{c_1c_2})\mathrm{D_x} + \mathrm{CD_x^3}]\mathrm{G'G} = 0 \qquad (\mathbf{23})$$

4. Infinite conservation law for the damped force Gardner equation

With the help of $V = V_1 - V_3 = V_4 - V_2$, eliminating V_3 and V_4 from the Eq.17 and combining them, we get

$$V_{1,x} - V_{2x} + \mathrm{V_{1,x}^2} - V_x^2 = \mathrm{c_1} c_2 \qquad (24)$$

Setting $c_1 = c_2 = \epsilon$ and putting $V_{1,x} = \sum_{n=1}^{\infty} I_n \epsilon^{-n} + \epsilon$ in Eq.24 yields

$$.\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} I_k I_{n-k} \epsilon^{-n} + \sum_{n=1}^{\infty} I_{n,x}\, \epsilon^{-n} + \sum_{n=1}^{\infty} I_{n+1}\, \epsilon^{-n} + 2I_1 - V_{2x} - V_x^2 = 0 \qquad (25)$$

After collecting the coefficients of different power of ϵ, we have a recursion relation for the conserved densities I_n

$$I_1 = \frac{1}{2}\left(\frac{H_x}{s_0 e^{-Lt}} + \left(\frac{H - e^{-Lt}\int e^{Lt}\Delta(t)dt}{s_0 e^{-Lt}}\right)^2\right)$$

$$I_2 = -\frac{1}{2}\left(\frac{H_{2x}}{s_0 e^{-Lt}} + 2\left(\frac{H - e^{-Lt}\int e^{Lt}\Delta(t)dt}{s_0 e^{-Lt}}\right)^1 \frac{H_x}{s_0 e^{-Lt}}\right)$$

$$\ldots\ldots$$

$$I_n = \sum_{k=1}^{n-1} I_k I_{n-k} + I_{n,x} + I_{n+1} \qquad (26)$$

where n=2, 3, 4, ⋯.

Again combining Eq.18 and Eq.19, we get

$$(V_2 - V_1)_t + (p(t) + 3c_1c_2C)(V_2 - V_1)_x + C(Y_{3x}(V_2, W_2) - Y_{3x}(V_1, W_1)) = 0 \quad (27)$$

Here on simplification, it could be expressed as

$$\frac{\partial}{\partial t}(V_1 - V)_x + \frac{\partial}{\partial x}\left[p(t)(V_1 - V)_x + C\left(6c_1c_2V_{1,x} + 2V_x^3 - 2V_{1,x}^3 + V_{1,3x} - V_{3x}\right)\right] = 0 \quad (28)$$

Setting $c_1 = c_2 = \epsilon$ and Eq. 28 provides

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} I_n\epsilon^{-n} - V_x\right) + \frac{\partial}{\partial x}\begin{bmatrix} p(t)\left(\sum_{n=1}^{\infty} I_n\epsilon^{-n} - V_x\right) + C\left(-2\sum_{n=3}^{\infty}\sum_{i+j+k=n} I_iI_jI_k\,\epsilon^{-n}\right) \\ -6\sum_{n=1}^{\infty}\sum_{k=1}^{n} I_kI_{n+1-k}\,\epsilon^{-n} + \sum_{n=1}^{\infty} I_{n,2x}\epsilon^{-n} + 2V_x^3 - V_{3x}\end{bmatrix} = 0 \quad (29)$$

which yields an infinite number of conservation laws.

$$\frac{\partial}{\partial t}Q_n + \frac{\partial}{\partial x}R_n = 0 \quad (30)$$

Here conserved densities Q_n are given by the Eq. 26 and the conserved fluxes R_n can be expressed as recursion relation

$$Q_0 = \frac{H - e^{-Lt}\int e^{Lt}\Delta(t)dt}{s_0e^{-Lt}}$$

$$R_0 = -\left(A + Be^{-Lt}\int e^{Lt}\Delta(t)dt\right)\left(e^{-Lt}\int e^{Lt}\Delta(t)dt\right)\frac{H - e^{-Lt}\int e^{Lt}\Delta(t)dt}{s_0e^{-Lt}} + C\left[2\left(\frac{H - e^{-Lt}\int e^{Lt}\Delta(t)dt}{s_0e^{-Lt}}\right)^3 - \frac{H_{2x}}{s_0e^{-Lt}}\right]$$

$$Q_1 = I_1$$

$$R_1 = \left(A + Be^{-Lt}\int e^{Lt}\Delta(t)dt\right)\left(e^{-Lt}\int e^{Lt}\Delta(t)dt\right)I_1 + C\left(-6I_1^2 + I_{1,2x}\right)$$

$$\cdots\cdots\cdots\cdots$$

$$Q_n = I_n$$

$$R_n = \left(A + Be^{-Lt}\int e^{Lt}\Delta(t)dt\right)\left(e^{-Lt}\int e^{Lt}\Delta(t)dt\right)I_n + C\left(-2\sum_{i+j+k=n} I_iI_jI_k - 6\sum_{k=1}^{n} I_kI_{n+1-k} + I_{n,2x}\right) \quad (31)$$

with $n = 3, 4, 5, \cdots$.

The Eq. 26 and Eq.31 showed the recursion relation for producing infinite number of conservation laws and the first conservation law gives exactly the df Gardner equation i.e. $\frac{\partial}{\partial t}Q_0 + \frac{\partial}{\partial x}R_0$ gives the df Gardner equation

$$H_t + AHH_x + BH^2Hx + CH_{xxx} + LH = \Delta(t) \quad (32)$$

5. Conclusion

We see that many infinite conservation laws of the Gardner equation in present of both damped term and force do exist that help to indicate the completely integrability of the equation. The Eq.30 and Eq.31 showed that the conserved densities Q_n and R_n are local and and obviously they depend on the constant coefficients A, B, C and damping coefficients L and external forcing term $\Lambda(t)$. Here the infinite conservation laws has been obtained based on the Bäcklund transformation i.e. Eq.17 and Eq.18.It is noted that Eq.17 provides a riccaty type equation and Eq.18 gives a divergence type equation which provides conserved densities and conserved fluxes respectively.

The Bell polynomials play a crucial role in characterization of bilinear Bäcklund transfor- mation and infinite conservation laws. Thus it may be assumed that there may be a deep relation of Bell polynomials with integrable frameworks of non-linear partial differential equation which remain open like the connection of Bell polynomial with symmetries, Hamiltonian function etc.

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