A STUDY OF DIFFERENTIAL EQUATIONS IN ENGINEERING APPLICATIONS

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Abstract: A second order differential equation is a particular kind of differential equation in which the function's derivative is of order 2, and no further derivatives of the function are included in the equation. It contains terms that denote the function's second order derivative, such as y", d2y/dx2, y"(x), etc. A second order differential equation is typically expressed as follows: y" + p(x)y' + q(x)y = f(x), where p(x), q(x), and f(x) are functions of x. Using the auxiliary equation and other techniques, including the method of unknown coefficients and parameter modification, we may solve this differential equation. If the functions p(x) and q(x) are constants, the differential equation y" + p(x)y' + q(x)y = 0 is referred to as a second-order differential equation with constant coefficients; if they are not, it is referred to as a second-order differential equation with variable coefficients. We shall examine these differential equations in-depth as well as their various varieties in this post. Along with learning alternative techniques, we will also learn how to determine the auxiliary equation and solve examples of second order differential equations with constant coefficients.

***Index Terms* – Order, method, differential equations, coefficients, variables.**

I. INTRODUCTION

Mathematical equations involving functions and all the associated derivatives are known as differential equations. They display a function's variability for a specified independent variable. When it comes to systems and procedures involving rates of change, these equations serve as the cornerstone. The order of the largest derivative of a differential equation is used to categorize the equations.

An equation containing one or more terms and the derivatives of one variable—the dependent variable—with respect to the other variable is called a differential equation (i.e., independent variable).

$$f(x) = dy/dx$$

In this case, the independent variable is "x," while the dependent variable is "y."
For instance, 5x = dy/dx

There are two types of derivatives in differential equations: partial and ordinary derivatives. The differential equation explains a relationship between a quantity that is continuously varying and the change in another quantity. The derivative is a measure of change rate. Numerous formulas for solving differential equations exist to determine the derivatives.

The highest order derivative included in the differential equation determines its order. Here are a few instances of the differential equation for various orders.

- dy/dx = 7x + 8. The equation has the following order: 1
- $(d^2y/dx^2) + 9\ (dy/dx) + y = 0$. It is in the order 2
- (dy/dt)+y = ct. The order is 1

II. SECOND ORDER DIFFERENTIAL EQUATIONS

A certain type of differential equation known as second-order differential equations has the second derivative as the greatest involved derivative. These comprise the depiction of a physical system displaying acceleration, such as vibrations, motion, and oscillations. Second-order equations require more complex formulas than first-order equations do, and typically require the initial conditions of motion in order to determine the solutions.

General Form for Differential Equations in Second Order.
A second-order differential equation can be expressed in general form as follows:

$$a.d^2y/dx^2 + b.dy/dx + cy = f(x)$$

when, f(x) is a function of x, where a, b, and c are constants.

Categories of Differential Equations in Second Order :
Different forms of second-order differential equations can be distinguished according to their attributes and features. Typical varieties include some of the following:

- Equations that are homogeneous and non-homogeneous
- Differential Equations: Linear and Non-Linear
- Differential Equation of Second Order with Constant Coefficients
- Differential Equation of Second Order with Variable Coefficients

Equations that are homogeneous and non-homogeneous:
Equations that are homogeneous take the form $a.d^2y/dx^2 + b.dy/dx + cy = 0$. As an illustration, consider:

$$8.d^2y/dx^2 + 9.dy/dx + y = 0$$

Equations that are non-homogeneous have an extra function, like f(x), on the right side, which makes them:

$$f(x) = a.d^2y/dx^2 + b.dy/dx + cy$$

Differential Equations: Linear and Non-Linear:
Equations with a dependent variable and its derivatives appearing linearly are known as linear differential equations. The equation dy/dx+P(x)y = Q(x), where P(x) and Q(x) are functions of x, can be used to express them. As an illustration, consider:

$$dy/dx + 2y = \sin(x)$$

Conversely, non-linear differential equations are those in which there is a non-linear appearance of the dependant variable and its derivatives. These equations can have a variety of solutions, sometimes displaying chaotic behaviour, and they do not adhere to the superposition principle. A typical illustration is:

$$dt/dy = y^2$$

Differential Equation of Second Order with Constant Coefficients:
Equations with constant coefficients, such as d2y/dx2 + 2dy/dx + 3y = 0, have coefficients that remain constant throughout the equation. The coefficients in equations with variable coefficients change depending on the independent variable.
As an illustration,

$$2y'' - 8y' + 6y = \cos x;$$
$$3y'' + 6y' - 9y = x$$

Differential Equation of Second Order with Variable Coefficients:
Variable coefficient differential equations are Second Order Differential Equations where the differential equation's coefficient is a variable.
As an illustration,

$$\ln(x)y'' - xy' + y = \cos x$$
$$xy'' + y' - e^xy = x$$

III. SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

Second-order differential equation solutions look like this:

- ***Both a general and specific solution***

A second-order differential equation's general solution consists of the particular integral (non-homogeneous solution) and the complementary function (homogeneous solution).
For instance, $y = c_1e^{-2x} + c_2xe^{-2x} + 2$ is the general solution for $d^2y/dx^2 + 2\ dy/dx + 8y = 4$.

- ***Problems with Initial Value and Boundary Value***

Finding a solution to an equation that meets beginning conditions, such as y(0) = 1 and dy/dx(0) = 2, is the goal of initial value problems. Finding a solution to boundary value problems, such as y(0) = 1, y(1) = 3, that satisfies the equation and specified conditions at various locations is the goal.

IV. SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

Different types for Solving Second-Order Differential Equations are:

- Analytical Method
- Numerical Method

Analytical Method :

Differential equations are solved analytically using techniques like the change of parameters approach and indeterminate coefficients. For an undetermined coefficients technique, the specific solution is approximated in a specific form, and the coefficients are obtained by re-entering the solution into the differential equation.

To demonstrate how to solve second-order linear differential equations with constant coefficients using the indeterminate coefficients approach, let's look at the following example:

Example: Solve the differential equation $y'' - 3y' + 2y = 2e^x$

Solution:

Step 1: *Solve the associated homogeneous equation*

$y'' - 3y' + 2y = 0$

Characteristic equation is $r^2 - 3r + 2 = 0$, *which has roots* $r_1 = 1$ *and* $r_2 = 2$

Therefore, the general solution of the homogeneous equation is:

$y_c = c_1e^x + c_2e^{2x}$

Step 2: *Assume a particular solution of the form* $y_p = Ae^x$

Substitute y_p *into the original equation:*

$(Ae^x)'' - 3(Ae^x)' + 2(Ae^x) = 2e^x$

Simplify and solve for A:

$Ae^x - 3Ae^x + 2Ae^x = 2e^x$

$-Ae^x = 2e^x$

$A = -2$

Step 3: *Write the general solution*

$y = y_c + y_p$

$y = c_1e^x + c_2e^{2x} - 2e^x$

Numerical Method :

When an analytical solution is not feasible, numerical approaches are applied. Calculating approximate solutions involves using techniques like Runge's Kutta Methods and Euler's Technique. The Euler Method uses a method of domain partitioning, to which tangent lines are subsequently applied in order to approximate the finite solution.

Higher-order numerical approaches called Runge-Kutta Methods provide a specific kind of approximation called a higher-order Runge-Kutta by utilising several intermediary steps.

To further understand Euler's approach to solving second-order differential equations, let's look at the following example:

Example:

Solve the initial value problem y′ = x + y, y(0) = 1 on the interval using Euler’s method with step size h = 0.25.

Solution:

Step 1: *Discretize the interval using the step size h = 0.25:*

$x_0 = 0, x_1 = 0.25, x_2 = 0.50, x_3 = 0.75, x_4 = 1.00$

Step 2: *Use Euler’s method to approximate the solution at each point:*

$y_n+1 = y_n + hf(x_n, y_n)$

Where $f(x, y) = x + y$

$y_0 = 1$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25(0 + 1) = 1.25$

$y_2 = y_1 + hf(x_1, y_1) = 1 .25 + 0.25(0.25 + 1.25) = 1.625$

$y_3 = y_2 + hf(x_2, y_2) = 1.625 + 0.25(0.50 + 1.625) = 2.125$

$y_4 = y_3 + hf(x_3, y_3) = 2.125 + 0.25(0.75 + 2.125) = 2.75$

Therefore, using Euler's method with a step size of h = 0.25, the approximate solution at x = 1 is y ≈ 2.75.

3.1 Solving Homogeneous Second Order Differential equations:

We can take the following actions to solve a homogeneous second-order differential equation of the type $a\, d^2y/dx^2 + b\, dy/dx + cy = 0$, where a, b, and c are constants:

Simplify the equation and replace $y = e^{rx}$ to find the characteristic equation.

- Determine the roots r_1 and r_2 by solving the characteristic equation.
- $Y = c_1e^{r1x} + c_2e^{r2x}$ is the universal solution if the roots are genuine and distinct.
- The usual solution, if the roots are complex conjugates, is $y = e^{\alpha xv}\{c_1\cos(\beta x) + c_2\sin(\beta x)]$, where α and β are found from the roots.
- The general solution, where r is the repeated root, is $y = (c_1+c_2x)e^{rx}$ if the roots are repeated.

3.2 Solving Non-Homogeneous Second-Order Differential Equations

The method of undetermined coefficients or the method of variation of parameters can be used to solve a non-homogeneous second-order differential equation of the form $a\, d^2y/dx^2 + b\, dy/dx + cy = f(x)$, where a, b, and c are constants and f(x) is a function of x.

- To determine the general answer, y_c, solve the related homogeneous equation.
- Using the method of unknown coefficients or modification of parameters, find a specific solution y_p.
- The non-homogeneous equation has a general solution of $y = y_c + y_p$.

IV. Examples of Second Order Differential Equation

In many different domains, complex systems are modelled and analysed using second-order differential equations. Here are some instances of the various fields in which second-order differential equations are used:

1. Vibrations in the Mechanical Domain:

Second-order differential equations are used in mechanical systems to simulate the motion of masses attached to springs and dampers. For instance, the mass-spring-damper system's equation of motion is:

$$F(t) = m\, d^2x/dt^2 + c\, dx/dt + kx$$

Mechanical systems, like car suspension systems, can have their vibrations and oscillations examined using this equation.

2. Electrical Systems:

Second-order differential equations are used in electrical engineering to examine the behaviour of resistor-inductor-capacitor (RLC) circuits. In an RLC series circuit, the current i(t) is determined by the following equation:

$$V(t) = L\, d^2i/dt^2 + R\, di/dt + 1/C\, i$$

Electrical circuits' transient and steady-state behaviour, including how they react to step inputs and sinusoidal excitations, can be studied using this equation.

3. Structural Analysis:

Second-order differential equations are used in civil and structural engineering to examine how structures behave under varied loads, including wind, seismic activity, and impact forces. For instance, the following is the equation of motion for a structure with
one degree of freedom that is accelerated by the ground:

$$c\, dx/dt + kx + m\, d^2x/dt^2 = -m_x g$$

This formula can be used to develop suitable structural systems and evaluate a building's seismic reaction.

V. Applications

Differential equations find several uses in various domains, including science, engineering, and applied mathematics.
In addition to their technical uses, they are employed in the resolution of several issues in everyday life. Let's look at some real-time differential equation applications.

Differential equations are used to explain different types of exponential growth and decay.

2) They're used to explain how return on investment varies over time.

3) They are employed in the study of medicine to simulate the development of cancer or the dissemination of illness within the body.

4) It can also be used to explain how electricity moves.

5) They assist economists in determining the best ways to make investments.

6) These equations can also be used to describe the motion of a pendulum or waves.

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