A NOTE ON INFINITE CONTINUED FRACTIONS AND RATIONAL APPROXIMATION

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Abstract: *We have seen that all rational numbers, can be represented as finite simple continued fractions. The main reason of interest of continued fractions, however, is in their application to the representation of irrational numbers. In this article, we shall show that every irrational number can be expressed as an infinite continued fraction*

Keywords: Infinite continued fraction, rational and irrational numbers, rational approximation

2. Infinite continued fraction

To expand an irrational number, we need infinite continued fractions; for example

$$\sqrt{2}+1 = 2+\left(\sqrt{2}-1\right) = 2+\frac{1}{\sqrt{2}+1} = 2+\cfrac{1}{2+\cfrac{1}{\sqrt{2}+1}}$$

$$= 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\sqrt{2}+1}}} = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}$$

The expression of $\sqrt{2}+1$ as a continued fraction uncovers a remarkable elegance and regularity, as opposed to its decimal representation, which does not show any regularity.

Definition 1: Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, all positive except possibly a_0. Infinite continued fraction is denoted by $[a_0; a_1, a_2, \dots]$. The infinite continued fraction is said to converge if the limit $\lim_{n\to\infty}[a_0; a_1, a_2, \dots, a_n]$ exists, and in that case the limit is also denoted by $[a_0; a_1, a_2, \dots]$.

We know that, $[a_0; a_1, a_2,\dots,a_n] = C_n$, the above limit can be written as $\lim_{n\to\infty}[a_0; a_1, a_2, \dots, a_n] = \lim_{n\to\infty} C_n$. Let us now existence of the above limit. By Theorem 3, we have $C_0 < C_2 < \dots < C_{2i} < \dots < C_{2j+i} < \dots < C_3 < C_1$. Because the even-numbered convergents C_{2n} form monotonically increasing sequence and bounded above by C_1, they will converge to a limit α that is greater than each C_{2n}. Similarly, odd numbered convergents C_{2n+1} are monotonically decreasing and bounded below by C_0 and hence converges to α' that is less than each C_{2n+1}. Let us prove $\alpha = \alpha'$. We have

$$p_{2n+1}q_{2n} - p_{2n}q_{2n+1} = (-1)^{2n} = 1.$$

Consider,

$$\alpha' - \alpha < C_{2n+1} - C_{2n} = \frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}} = \frac{1}{q_{2n}q_{2n+1}}$$

and hence

$$0 \leq |\alpha' - \alpha| < \frac{1}{q_{2n}q_{2n+1}} < \frac{1}{q_{2n}^2}.$$

Since the q_i increases as i becomes large, $\frac{1}{q_{2n}^2} \to 0$ as $n \to \infty$. Hence $\alpha = \alpha'$.

Theorem 1: The value of any infinite continued fraction is an irrational number.

Proof: Let us suppose that x denotes the value of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$; that is, x is the limit of the sequence of convergents

$$C_n = [a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}.$$

Because x lies strictly between the successive convergents C_n and C_{n+1}, we have

$$0 < |x - C_n| < |C_{n+1} - C_n| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}}.$$

With the view to obtaining a contradiction, assume that x is a rational number, say, $x = a / b$, where a and b > 0 are integers. Then

$$\left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

and so,

$$|aq_n - bp_n| < \frac{b}{q_{n+1}}.$$

As q_i increase without bounds as i increases, we can chose n so large that b < qn+1 and hence

$$0 < |aq_n - bp_n| < 1.$$

This shows that there is a positive integer between 0 and 1, which is a contradiction.

The converse of the above theorem is also true.

Theorem 2: Every irrational number has a unique representation as an infinite continued fraction.

Proof: Let x_0 be an arbitrary irrational number. Let us find the sequence of integers $a_0, a_1, a_2, \ldots$ as follows: Let

$$a_k = [x_k] \text{ and } x_{k+1} = \frac{1}{x_k - a_k} k \geq 0 \qquad (1)$$

It is evident that x_{k+1} is irrational whenever x_k is irrational. Since x_0 is irrational all x_k are irrational by induction. Thus,

$$0 < x_k - a_k = x_k - [x_k] < 1 \qquad (2)$$

and hence

$$x_{k+1} = \frac{1}{x_k - a_k} > 1 \qquad (3)$$

so that the integers $a_{k+1} = [x_{k+1}] \geq 1$ for all $k \geq 0$. Thus, we have a sequence of integers $a_0, a_1, a_2, \ldots$, all positive except perhaps for a_0.

Now, (3) can be written as

$$x_k = a_k + \frac{1}{x_{k+1}},\ k \geq 0.$$

Through successive substitutions, we obtain

$$\begin{aligned} x_k &= a_0 + \frac{1}{x_1} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}} \\ &\vdots \\ &= [a_0; a_1, a_2, \cdots, x_{n+1}] \end{aligned}$$

for every positive integer n. Now, we have to prove that the infinite simple continued fraction $[a_0, a_1, a_2, \ldots]$ indeed converges to x_0.

Let n be a fixed positive integer. Then,

$$x_0 = [a_0; a_1, a_2, \cdots, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}$$

where $C_n = \frac{p_n}{q_n}$ is the n^{th} convergent of $x_0 = [a_0; a_1, a_2, \cdots]$. Hence,

$$\begin{aligned} x_0 - C_n &= \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{-(p_nq_{n-1} - p_{n-1}q_n)}{(x_{n+1}q_n + q_{n-1})q_n} \\ &= \frac{-(-1)^{n-1}}{(x_{n+1}q_n + q_{n-1})q_n} \end{aligned}$$

From (2), we have $x_{n+1} > a_{n+1}$ and therefore

$$|x_0 - C_n| = \frac{1}{(x_{n+1}q_n + q_{n-1})q_n} < \frac{1}{(a_{n+1}q_n + q_{n-1})q_n} = \frac{1}{q_{n+1}q_n}$$

Because qk increases without bounds as k increases, $\frac{1}{q_{n+1}q_n} \to 0$ as $n \to \infty$. Hence

$$x_0 = \lim_{n \to \infty} C_n = [a_0; a_1, a_2, \cdots].$$

Example 1: Consider the irrational number $x_0 = \sqrt{23}$. The successive irrational numbers x_k (and hence a_k) can be computed as follows:

$$x_0 = \sqrt{23} = 4 + \sqrt{23} - 4 \qquad [\because [\sqrt{23}] = 4 \qquad a_0 = 4$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7} \qquad a_1 = 1$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{7}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{2} = 3 + \frac{\sqrt{23} - 3}{2} \qquad a_2 = 3$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{2}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{7} = 1 + \frac{\sqrt{23} - 4}{7} \qquad a_3 = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{\sqrt{23} - 4} = \sqrt{23} + 4 = 8 + (\sqrt{23} - 4) \qquad a_4 = 8$$

Because $x_5 = x_1$, also $x_6 = x_2$, $x_7 = x_3$, $x_8 = x_4$; then we get $x_9 = x_5 = x_1$, and so on, which means that the block of integers 1, 3, 1, 8 repeats indefinitely. We find that the continued fraction expansion of $\sqrt{23}$ is periodic with the form

$$\sqrt{23} = [4;1, 3, 1, 8, 1, 3, 1, 8, \cdots]$$
$$= [4;\overline{1, 3, 1, 8}]$$

Now, we prove that the representation of an irrational number as an infinite continued fraction is unique in the following theorem.

Theorem 3: If the two infinite simple continued fractions $[a_0;a_1,a_2,\cdots]$ and $[b_0;b_1,b_2,\cdots]$ represent the same irrational number x, then $a_k = b_k$ for k = 0, 1, 2, 3, …

Proof: Suppose that $x = [a_0;a_1,a_2,\cdots]$. Then, $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1}$ we have from Theorem 4 of Chapter 15,

$a_0 < x < a_0 + \frac{1}{a_1}$ so that $a_0 = [x]$. Note that

$$[a_0;a_1,a_2,\cdots] = a_0 + \frac{1}{[a_1;a_2,a_3,\cdots]}$$

Suppose that $[a_0;a_1,a_2,\cdots]$ = $[b_0;b_1,b_2,\cdots]$ then clearly, $a_0 = b_0 = [x]$ and that

$$a_0 + \frac{1}{[a_1;a_2,a_3,\cdots]} = b_0 + \frac{1}{[b_1;b_2,b_3,\cdots]}$$

so that

$$[a_1;a_2,a_3,\cdots] = [b_1;b_2,b_3,\cdots]$$

Now assume that $a_k = b_k$ and that $[a_{k+1};a_{k+2},a_{k+3},\cdots] = [b_{k+1};b_{k+2},b_{k+3},\cdots]$. Using the same argument, we see that $a_{k+1} = b_{k+1}$, and

$$a_{k+1} + \frac{1}{[a_{k+2};a_{k+3},\cdots]} = b_{k+1} + \frac{1}{[b_{k+2};b_{k+3},\cdots]}$$

which implies

$$[a_{k+2};a_{k+3},\cdots] = [b_{k+2};b_{k+3},\cdots]$$

Hence by induction, we see that $a_k = b_k$ for k = 0, 1, 2, …

Theorem 4: If x is an irrational number, then there are infinitely many rational numbers p / q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} \tag{4}$$

Proof: Let p_k / q_k be the k^{th} convergent of the continued fraction of x. Then, by Theorem 2, we know that

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2} \qquad [\because q_k < q_{k+1}$$

Hence

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

Consequently, the convergents of x, p_k / q_k, $k = 1, 2, \ldots$ are infinitely many rational numbers which satisfy (4).

3. Rational approximation to irrational numbers

The following theorem and corollary shows that the convergents of the simple infinite continued fraction of an irrational numbers x are the best rational approximation to x.

Theorem 5: Let p_n / q_n be the n^{th} convergent of the continued fraction representing the irrational number x. If a and b are integers, with $1 \le b < q_n+1$, then

$$|q_n x - p_n| \le |bx - a|$$

Proof: Consider the system of equations

$$p_n \alpha + p_{n+1} \beta = a$$
$$q_n \alpha + q_{n+1} \beta = b$$

Then, the solutions of the above system of equations are given by

$$\alpha = (-1)^{n+1}(aq_{n+1} - bp_{n+1})$$
$$\beta = (-1)^{n+1}(bp_n - aq_n)$$

Note that $\alpha \ne 0$. For, if $\alpha = 0$, then $(aq_{n+1} = bp_{n+1})$ and, because $\gcd(p_{n+1}, q_{n+1}) = 1$, $q_{n+1} \mid b$ or $b \ge q_{n+1}$, which is a contradiction to our hypothesis.

If $\beta = 0$, then $a = p_n \alpha$ and $b = q_n \alpha$ and hence $|bx - a| = |\alpha||q_n x - p_n| \ge |q_n x - p_n|$, which is the required result. So, assume $\beta \ne 0$.

If $\beta < 0$, then the equation $q_n \alpha = b - q_{n+1}\beta$ implies that $q_n \alpha > 0$ and therefore $\alpha > 0$. If $\beta > 0$, then $b < q_{n+1}$ which implies $b < \beta q_{n+1}$ and therefore $\alpha q_n = b - q_{n+1}\beta < 0$; this makes $\alpha < 0$. Hence, α and β must have opposite signs. By Theorem 4 of Chapter 15, since x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$,

$$q_n x - p_n \qquad \text{and} \qquad q_{n+1} x - p_{n+1}$$

will have opposite signs. This implies

$$\alpha(q_n x - p_n) \qquad \text{and} \qquad \beta(q_{n+1} x - p_{n+1})$$

must have the same sign and therefore

$$\left|\alpha\left(q_n x - p_n\right) + \beta\left(q_{n+1}x - p_{n+1}\right)\right| = |\alpha||q_n x - p_n| + |\beta||q_{n+1}x - p_{n+1}|$$

Now, consider

$$\begin{aligned}|bx - a| &= |(q_n\alpha + q_{n+1}\beta)x - (p_n\alpha + p_{n+1}\beta)| \\ &= \left|\alpha\left(q_n x - p_n\right) + \beta\left(q_{n+1}x - p_{n+1}\right)\right| \\ &= |\alpha||q_n x - p_n| + |\beta||q_{n+1}x - p_{n+1}| \\ &> |\alpha||q_n x - p_n| \\ &> |q_n x - p_n|\end{aligned}$$

which is the desired inequality.

Corollary 1: If $1 \le b \le qn$, the rational number a / b satisfies

$$\left|x - \frac{p_n}{q_n}\right| < \left|x - \frac{a}{b}\right|$$

Proof: Suppose

$$\left|x - \frac{p_n}{q_n}\right| > \left|x - \frac{a}{b}\right|$$

then

$$|q_n x - p_n| = q_n\left|x - \frac{p_n}{q_n}\right| > b\left|x - \frac{a}{b}\right| = |bx - a|$$

which is a contradiction to Theorem 5.

Theorem 6: Let x be an arbitrary irrational number. If the rational number a / b where $b \ge 1$ and $\gcd(a, b) = 1$, satisfies

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$$

then a / b is one of the convergents p_n / q_n in the continued fraction representation of x.

Proof: Assume that a / b is not a convergent of x. Since the sequence q_n is an increasing sequence, there exists a unique integer n for which $q_n \le b < q_n+1$. For this n, the last lemma gives the first inequality in the chain

$$|q_n x - p_n| = |bx - a| = b\left|x - \frac{a}{b}\right| < \frac{1}{2b}$$

which may be written as

$$\left|x - \frac{p_n}{q_n}\right| > \frac{1}{2bq_n}$$

Since, $a / b \ne p_n / q_n$, $bp_n - aq_n$ is a nonzero integer, and hence $1 \le |bp_n - aq_n|$.

Now, consider

$$\frac{1}{bq_n} \le \left|\frac{bp_n - aq_n}{bq_n}\right| = \left|\frac{p_n}{q_n} - \frac{a}{b}\right| \le \left|\frac{p_n}{q_n} - x\right| + \left|x - \frac{a}{b}\right| < \frac{1}{2bq_n} + \frac{1}{2b^2}$$

Since, $q_n \ge b$,

$$\frac{1}{2bq_n}+\frac{1}{2b^2}\le\frac{1}{2b^2}+\frac{1}{2b^2}=\frac{1}{b^2}.$$

Therefore

$$\frac{1}{bq_n}<\frac{1}{b^2}\Rightarrow\frac{1}{q_n}<\frac{1}{b}\Rightarrow q_n>b.$$

But this is a contradiction to the fact that $q_n \le b$. This completes the proof.

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