Oscillation of Fourth-Order Nonlinear Semi-canonical Difference Equations

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Abstract

This paper presents the oscillation criteria for the equation

$$D\left(a_3(n)\Delta\left(a_2(n)\Delta(a_1(n)\Delta y(n))\right)\right) + q(n)y^{\alpha}(n-k) = 0, (E)$$

where $\sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} < \infty, \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty$ and $\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty$

First, we transform the equation (E) into a canonical type equation, which reduced the set of nonoscillatory solution of (E) into two types instead of six. Therefore, the investigation of finding oscillation criteria become easy since we have to eliminate only two set of nonoscillatory solutions. Hence the results obtained hence are new complement to existing result.

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1. Introduction

In this paper, we investigate the oscillatory behavior of the semi-canonical fourthorder delay difference equation of the form

$$D_4 y(n) + q(n)y^{\alpha}(n-k) = 0, n \geq n_0 \geq 1$$

where $D_0 y(n) = y(n), D_i y(n) = a_i(n)\Delta(D_{i-1}y(n)), i = 1,2,3$ and $D_4 y(n) = \Delta(D_3 y(n))$, subject to the following conditions:
$(H_1)\{a_j(n)\}, j = 1,2,3$ are positive sequences of real numbers for all $n \geq n_0$;
$(H_2)\{q(n)\}$ is a positive real sequence for all $n \geq n_0$;
$(H_3)\alpha$ is a ratio of odd positive integers and k is a positive integer;
$(H_4)\Delta$ is the forward difference operator defined for $\Delta y(n) = y(n+1) - y(n)$.

By a solution of (E), we mean a real sequence $\{y(n)\}$ satisfying (E) for all $n \geq n_0$. We consider only such solutions there are nontrivial for all large n.

A solution of (E) is called nonoscillatory if it is eventually positive or eventually negative; otherwise it is called oscillatory.

Most of the results known for equation (E) when it is in canonical form, that is,

$$\sum_{n=n_0}^{\infty} \frac{1}{a_j(n)} = \infty, j = 1,2,3$$

Next, we introduce the following terminology and classification of (E). Define

$$A_i(n) = \sum_{s=n}^{\infty} \frac{1}{a_j(s)} = \infty, j = 1,2,3$$

(E) is canonical form if

$$A_i(n_0) = \infty, i = 1,2,3 \tag{1.1}$$

and in Noncanonical form if

$$A_i(n_0) < \infty, i = 1,2,3 \tag{1.2}$$

Semi-canonical form

$$A_1(n_0) = A_2(n_0) = \infty, A_3(n_0) < \infty \tag{1.3}$$

or

$$A_1(n_0) = \infty, A_2(n_0) < \infty, A_3(n_0) = \infty \tag{1.4}$$

or

$$A_1(n_0) < \infty, A_2(n_0) = \infty, A_3(n_0) = \infty \tag{1.5}$$

Finally, it is in semi-noncanonical form

$$A_1(n_0) = \infty, A_2(n_0) < \infty, A_3(n_0) < \infty \tag{1.6}$$

or

$$A_1(n_0) < \infty, A_2(n_0) = \infty, A_3(n_0) < \infty \tag{1.7}$$

or

$$A_1(n_0) < \infty, A_2(n_0) < \infty, A_3(n_0) = \infty \tag{1.8}$$

Recently, the authors discussed the oscillatory behavior of (E) under condition (1.2). They obtained criteria for the oscillation of (E) by eliminating the following eight possible nonoscillatory solutions of (E).
(i) $D_1y(n) > 0, D_2y(n) > 0, D_3y(n) > 0, D_4y(n) < 0$,
(ii) $D_1y(n) > 0, D_2y(n) > 0, D_3y(n) < 0, D_4y(n) < 0$,
(iii) $D_1y(n) > 0, D_2y(n) < 0, D_3y(n) < 0, D_4y(n) < 0$,
(iv) $D_1y(n) > 0, D_2y(n) < 0, D_3y(n) > 0, D_4y(n) < 0$,
(v) $D_1y(n) < 0, D_2y(n) > 0, D_3y(n) > 0, D_4y(n) < 0$,
(vi) $D_1y(n) < 0, D_2y(n) < 0, D_3y(n) > 0, D_4y(n) < 0$,
(vii) $D_1y(n) < 0, D_2y(n) > 0, D_3y(n) < 0, D_4y(n) < 0$,
(viii) $D_1y(n) < 0, D_2y(n) < 0, D_3y(n) < 0, D_4y(n) < 0$.

In authors discussed the oscillatory behavior of (E) when the condition (1.3) or (1.4) hold and the authors studied similar properties of (E) when the condition (1.5) holds. If there two papers, the authors reduced the equation (E) into canonical form and then obtained the criteria for the oscillation of (E) by eliminating only two sets of nonoscillatory solutions.

Motivated by the above observations, in this paper we find criteria for the oscillation of all solutions of (E) when the condition (1.7) holds. This is achieved by reducing the equation (E) into canonical type and then applying comparison with first-order delay difference equations.

2. Main Results

Define

$$b_1(n) = a_1(n)A_1(n)A_1(n+1), b_2(n) = \frac{a_2(n)}{A_3(n)A_1(n+1)}$$
$$b_3(n) = a_3(n)A_3(n)A_3(n+1), Q(n) = A_3(n+1)q(n)A_1^{\alpha}(n-k)$$

and

$$z(n)=\frac{y(n)}{A_1(n)}$$

Theorem 2.1. Assume that

$$\sum_{n=n_0}^{\infty}\frac{A_3(n)A_1(n+1)}{a_2(n)}=\infty \tag{2.1}$$

Then the semi-canonical operator D_4y can be written in canonical form as:

$$D_4y(n)=\frac{1}{A_3(n+1)}\Delta\left(b_3(n)\Delta\left(b_2(n)\Delta\left(b_1(n)\Delta\left(\frac{y(n)}{A_1(n)}\right)\right)\right)\right) \tag{2.2}$$

Proof. Using the difference calculus one can get

$$\frac{1}{A_3(n+1)}\Delta\left(b_3(n)\Delta\left(b_2(n)\Delta\left(b_1(n)\Delta\left(\frac{y(n)}{A_1(n)}\right)\right)\right)\right)=D_4y(n)$$

Further

$$\sum_{n=n_0}^{\infty}\frac{1}{b_3(n)}=\sum_{n=n_0}^{\infty}\frac{1}{a_3(n)A_3(n)A_3(n+1)}=\sum_{n=n_0}^{\infty}\Delta\left(\frac{1}{A_3(n)}\right)$$
$$=\lim_{n\to\infty}\frac{1}{A_3(n)}-\frac{1}{A_3(n_0)}=\infty$$
$$\sum_{n=n_0}^{\infty}\frac{1}{b_1(n)}=\sum_{n=n_0}^{\infty}\frac{1}{a_1(n)A_1(n)A_1(n+1)}=\sum_{n=n_0}^{\infty}\Delta\left(\frac{1}{A_1(n)}\right)$$
$$=\lim_{n\to\infty}\frac{1}{A_1(n)}-\frac{1}{A_1(n_0)}=\infty$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{b_2(n)}=\sum_{n=n_0}^{\infty}\frac{A_1(n+1)A_3(n)}{a_2(n)}=\infty$$

by (2.1). Hence the right hand side of (2.2) in canonical form.
From the above theorem, we obtain the following Corollary.
Corollary 2.2. Let (2.1) hold. Then $y(n)$ is a solution of (E) if and only if the canonical equation

$$L_4z(n)+Q(n)z^{\alpha}(n-k)=0, n\geq n_0 \tag{c}$$

where $L_0z(n)=z(n), L_iz(n)=b_i(n)\Delta(L_{i-1}z(n)), i=1,2,3$ and $L_4z(n)=\Delta(L_3z(n))$, has a solution $z(n)=\frac{y(n)}{A_1(n)}$.

The next lemma gives us the classification of positive nonoscillatory solutions of (E_c). (see [1,5]).

Lemma 2.3. Let $\{z(n)\}$ be an eventually positive solution of (E_c). Then one of the following cases holds:
$(C_1)L_1z(n)>0, L_2z(n)<0, L_3z(n)>0, L_4z(n)<0$,
$(C_2)L_1z(n)>0, L_2z(n)>0, L_3z(n)>0, L_4z(n)<0$.
From the transformation $y(n)=A_1(n)z(n)$, we see that the oscillation of $z(n)$ implies that of $y(n)$. Therefore, the oscillation of (E_c) implies that of (E).

Theorem 2.4. Assume (2.1) holds. If

$$\sum_{n=n_0}^{\infty} Q(n) = \infty \tag{2.3}$$

then equation (E) is oscillatory.

Proof. Assume the contrary that $\{y(n)\}$ is an eventually positive solution of (E). Then by Corollary 2.2, the sequence $\{z(n)\}$ is also a positive solution of (E_c) and by Lemma 2.3, the sequence $\{z(n)\}$ belongs to (C_1) or (C_2), for all $n \geq n_1 \geq n_0$.

In both cases $z(n)$ is increasing, so there exists a constant $M > 0$ and an integer $n_2 \geq n_1$ such that $z(n) \geq M$ for all $n \geq n_2$. Using this in (E_c) and then summing from n_2 to n, we get

$$M^{\alpha} \sum_{s=n_2}^{n} Q(s) \leq L_3 z(n_2) - L_3 z(n+1) \leq L_3 z(n_2)$$

since in both cases $L_3 z(n) > 0$. This contradiction completes the proof.

Remark 2.5. This theorem is independent of the values of α and the delay argument k. Hence, it is applicable to linear, sublinear, or superlinear equations as well as to ordinary, delay or advanced type equations.

Note that the results true for canonical equations knows in the literature can be applied to the equation (E_c) to get oscillation results for (E).

Example 2.1. Consider the fourth-order delay difference equation

$$\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta(n(n+1)\Delta y(n))\right)\right) + q_0(n-k)^{\alpha} y^{\alpha}(n-k) = 0 \tag{2.4}$$

where $q_0 > 0$ and $n \geq n_0 + k$.
By a simple calculations, we see that $A_1(n) = A_3(n) = \frac{1}{n}, b_1(n) = 1, b_2(n) = n+1, b_3(n) = 1$ and $Q(n) = \frac{q_0}{n+1}$.
The condition (2.3) becomes

$$\sum_{n=n_0}^{\infty} \frac{q_0}{n+1} = \infty$$

that is, condition (2.3) holds if $q_0 > 0$. Therefore by Theorem 2.4, the equation (2.4) is oscillatory.

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