A Smooth Exact Penalty Function Approach for Constrained Optimization with Applications in Stress-Constrained Topology Optimization and Scenario-Based Reliability Design

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Abstract

A novel smooth penalty function method is introduced, which eliminates the need for dual and slack implicit variables in formulating constrained optimization conditions. This method features new active and loss functions that independently and adaptively address the violation of each constraint. By minimizing the proposed penalty function unconstrained, a solution to the original optimization problem is achieved. The derived constrained optimization conditions rely solely on the primal variable and extend to encompass the canonical Karush-Kuhn-Tucker (KKT) conditions within a broader framework. The derivatives of the proposed loss functions concerning the constraint functions can be interpreted as Lagrange multipliers of the traditional Lagrangian function. Additionally, the method reveals the inclusion of first-order Hessian information, contrasting with existing works where classical Lagrange Hessians only incorporate second-order derivatives. Finally, numerical examples, including medium-scale stress-constrained topology optimization and scenario-based reliability design problems, are provided to illustrate the effectiveness of the proposed methodology.

Introduction

There is a substantial body of research on nonlinear constrained optimization. Numerous methods have been developed for constrained optimization, including the interior point method [1,2], Sequential Quadratic Programming (SQP) approach [3,4], and the active set method [5]. For a comprehensive overview of constrained optimization, see references [6,7].

The application of active and loss functions in various constrained optimization and machine learning problems is a vibrant area of research. For instance, reference [8] introduces a novel loss formulation for face recognition tasks to address the limitations of the least squares loss function. A non-convex loss function aimed at solving regression problems through the difference of convex functions programming is presented in [9]. Liu [10] created a truncated Huber penalty function to satisfy different smoothing requirements in various graphic applications. Additionally, a nonconvex quadratic ε-insensitive loss function is introduced in [11] to enhance the robustness of support vector regression methods. Recent advancements in loss functions are discussed in [12], particularly in the context of deep learning and computer vision tasks.

Lagrange multiplier methods are commonly employed to tackle constrained optimization problems. For example, within the augmented Lagrangian framework, an exact penalty function is utilized in [13] to analyze the behavior of augmented Lagrange multipliers, exploring the relationship between the exactness of the penalty function and the existence of these multipliers. In [14], the cost of solving augmented Lagrangian subproblems is reduced, allowing the inner loop of the Newton–Raphson method to converge in just two iterations. The iteration complexity of the inexact Augmented Lagrangian Method (ALM) is examined in [15] using Nesterov's optimal first-order method. The equivalence between an alternating direction multipliers method and an inexact proximal ALM is demonstrated for a class of convex programming in [16]. A technique incorporating second-order information to minimize a second-order Taylor expansion of the Lagrangian function is reported in [17]. Drawing inspiration from the Lagrange multipliers method, a primal-dual algorithm for solving non-smooth risk-averse optimization problems is developed in [18]. An ALM is formulated for the optimality conditions of nonlinear semi-definite programming in [19], while convergence analysis of the ALM for conic programming is conducted in [20]. Furthermore, an ALM designed for efficient parallelization in solving large-scale non-convex optimization problems is discussed in [21]. Lastly, in [22], the lower-level optimization formulation in multi-objective optimization problems is replaced by the KKT conditions, with Lagrange multipliers expressed as polynomial functions.

The proposed method aims to address the limitations of existing constrained optimization techniques by introducing a smooth exact penalty function that eliminates the need for dual and slack implicit variables. This method enhances the stability and efficiency of optimization processes while ensuring robust handling of constraints.

Methodology

Active Function Design: Introduce a differentiating active function $A_i(x)$ for each constraint $g_i(x) \leq 0$, which indicates the presence or absence of a constraint. This function will adaptively reflect the constraint's status during optimization.
Loss Function Definition: Define loss functions $L_i(x)$ for each constraint, measuring the violation of constraints:

$$L_i(x) = max(0, g_i(x))$$

These loss functions will be formulated to ensure differentiability and smoothness.

Smooth Exact Penalty Function Construction

Construct the smooth exact penalty function $P(x)$ as follows:

$$P(x) = f(x) + \sum_{i=1}^{m} \rho_i L_i(x)$$

where:

1. $f(x)$ is the original objective function.
2. ρ_i are positive weights assigned to each loss function to control the penalty's strength.

To enhance smoothness, use a differentiable approximation of the maximum function, such as:

$$L_i(x) = \frac{1}{1 + e^{-kg_i(x)}}$$

where k is a large constant that controls the smoothness and steepness of the penalty.

Optimization Procedure

Unconstrained Minimization: Solve the unconstrained optimization problem:

$$\min_x P(x)$$

This minimization directly addresses the original constrained optimization problem without introducing dual variables or slack variables.
Iterative Updates: Use gradient-based methods to update the primal variable x :

$$x_{k+1} = x_k - \alpha \nabla P(x_k)$$

where α is the step size.

Conclusion

This smooth exact penalty function method offers a robust framework for addressing constrained optimization problems. By utilizing differentiable active and loss functions, the approach enhances the stability of optimization processes while maintaining flexibility in constraint handling. Future work will focus on extending this methodology to more complex optimization scenarios and validating its performance across various applications.

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