Oscillation of Noncanonical Second-Order Trinomial Difference Equations with Retarded and Advanced Arguments

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Abstract

In this paper, using a new technique we studied the oscillation of all solutions of the noncanonical second-order trinomial equation with retarded and advanced arguments $\Delta(a(n)\Delta y(n)) = p(n)y(\tau(n)) + q(n)y(\sigma(n))$.

The obtained criteria improve the existing ones and this is illustrated via some examples.

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1. Introduction

This paper deals with oscillatory behavior of a functional difference equation of the form

$$\Delta(a(n)\Delta y(n)) = p(n)y(\tau(n)) + q(n)y(\sigma(n)), n \geq n_0, \tag{E}$$

subject to the following conditions:
$(H_1)\{a(n)\}, \{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers with $a(n) > 0, p(n) > 0$ and $q(n) > 0$ for all $n \geq n_0$;
$(H_2)\{\tau(n)\}$ is an increasing sequence of integers with $\tau(n) \leq n - 1$ and $\lim_{n\to\infty} \tau(n) = \infty$;
$(H_3)\{\sigma(n)\}$ is an increasing sequence of integers with $\sigma(n) \geq n + 1$.

By a solution of equation (E), we mean a real sequence $\{y(n)\}$ defined and satisfies equation (E) for all $n \geq n_0$ and $\sup\{|y(n)|: n \geq N\} > 0$ for all $N \geq N_y$. Assume that (E) does possess such solutions. In the standard way, that is, a nontrivial solution is termed oscillatory or nonoscillatory according to whether it is neither eventually positive nor eventually negative or not.

In fact, oscillation and retarded/advanced phenomena appear in different models from real world applications. For models from mathematical biology and social sciences where oscillation and/or retarded/advanced actions may be formulated by means of cross-diffusion terms, see, for example, the papers [1-10]. There are many papers devoted to oscillation of difference equations, see, [7-10] and the references cited therein.

If $\{y(n)\}$ is a nonoscillatory solution of (E) then it satisfies one of the following two classes:
$S_0: y(n) > 0, \Delta y(n) < 0, \Delta(a(n)\Delta y(n)) > 0$,
$S_0: y(n) > 0, \Delta y(n) > 0, \Delta(a(n)\Delta y(n)) > 0$.
If we denote the st of all nonoscillatory (positive) solutions of (E) by S, then we have

$$S = S_0 \cup S_2. \tag{1.1}$$

It is clear that the corresponding difference inequalities

$$\Delta(a(n)\Delta y(n)) - p(n)y(\tau(n)) \geq 0 \tag{1.2}$$

and

$$\Delta(a(n)\Delta y(n)) - q(n)y(\tau(n)) \geq 0 \tag{1.3}$$

have the same set of nonoscillatory solution (1.1).

Recently in [8-10], the authors have shown that the classes S_0 and S_2 may empty for the case $\sigma(n) = n - k$ and $\sigma(n) = n + l$.

Theorem 1. If

$$\limsup_{n\to\infty} \sum_{s=n-k}^{n-1} \left(\frac{1}{a(s)} \sum_{t=s}^{n-1} p(t) \right) > 1 \tag{1.4}$$

then the set S_0 is empty for (1.2).

Theorem 2. If

$$\limsup_{n\to\infty} \sum_{s=n}^{n+l-1} \left(\frac{1}{a(s)} \sum_{t=n}^{s-1} q(t) \right) > 1 \tag{1.5}$$

then the set S_2 is empty for (1.3).

From the above results, we see that all solutions will be oscillatory for difference equations involving both advanced and retarded arguments. Hence, we are led to the following result.

Theorem 3. Assume that (1.4) and (1.5) hold. Then the equation

$$\Delta^2 y(n) = p(n)y(n - k) + q(n)y(n + l)$$

is oscillatory.

From the above theorems, we see that the criterion (1.4) does not contain the sequence $\{q(n)\}$ and similarly the information about the sequence $\{p(n)\}$ is lost in (1.5). Therefore the aim of this paper is to privide criteria for $S_0 = \phi$ and $S_2 = \phi$ that involve both sequences $\{p(n)\}$ and $\{q(n)\}$.

2. Main Results

In view of (H_2), we see that $\tau^{-1}(n)$ exists and there exists a sequence of integer $\{\xi(n)\}$ such that

$$\xi(\xi(n)) = \tau^{-1}(n) \tag{2.1}$$

is well defined.

Lemma 4. Assume that $\{\xi(n)\}$ satisfies (2.1). Then

$$\xi^{-1}(\xi^{-1}(n)) = \tau(n), \xi(n) = \tau^{-1}(\xi^{-1}(n)), \xi^{-1}(n) = \tau(\xi(n)) \tag{2.2}$$

Proof. The proof is very elementary and so it is omitted.

Our first result concerned with $S_0 = \phi$ for (E). For simplicity, we use the following notations:

$$A(n) = \sum_{s=n_0}^{n-1} \frac{1}{a(s)},$$

$$P_1(n) = \sum_{s=n}^{\xi(n)-1} (A(s+1) - A(n))p(s) \geq c_1 > 0,$$

$$P_3(n) = \sum_{s=\tau^{-1}(n)}^{\tau^{-1}(\xi(n))-1} (A(s+1) - A(n))p(s) \geq c_3 > 0,$$

and

$$Q_1(n) = \sum_{s=n}^{\xi(n)-1} (A(s+1) - A(n))q(s) \geq d_1 > 0$$

Also, we set

$$P_i^* = \frac{c_i}{1-c_2}, i = 1,3, Q_i^* = \frac{d_i}{1-c_2}, i = 1,2,$$

and

$$M = \frac{(P_1^*)^2}{1 - P_1^* P_3^*}, N = \frac{P_1^*}{1 - P_1^* P_3^* - P_1^* Q_1^* M}$$

Theorem 5. Assume that there exists a sequence $\{\xi(n)\}$ satisfying (2.1) and $\tau^{-1}(n) \geq \sigma(n)$. If

$$P_1^* N + MN(Q_1^* + P_1^* Q_2^*) + P_1^*[P_1^* + P_3^* + P_1^* Q_1^* N + Q_2^* MN] > 1 \tag{2.5}$$

then $S_0 = \phi$ for (E).

Proof. Let $\{y(n)\}$ be an eventually positive solution of (E) such that $y(n) \in S_0$. Summing up twice (E) from n to ∞ and then applying summation by parts formula, we have

$$y(n) = \sum_{s=n}^{\infty} (A(s+1) - A(n))p(s)y(\tau(s)) + (A(s+1) - A(n))q(s)y(\sigma(s))$$

Hence

$$y(n) \geq \sum_{s=n}^{\xi(n)-1} (A(s+1) - A(n))p(s)y(\tau(s)) + \sum_{s=\xi(n)}^{\tau^{-1}(n)-1} (A(s+1) - A(n))p(s)y(\tau(s))$$

$$+ \sum_{s=\tau^{-1}(n)}^{\tau^{-1}(\xi(n))-1} (A(s+1) - A(n))p(s)y(\tau(s)) + \sum_{s=n}^{\xi(n)-1} (A(s+1) - A(n))q(s)y(\tau^{-1}(s))$$

Using the fact that $\{y(n)\}$ is decreasing, we obtain

$$y(n) \geq C_1 y(\xi^{-1}(n)) + C_2 y(n) + C_3 y(\xi(n)) + d_1 y(\tau^{-1}(\xi(n))) + d_2 y(\tau^{-1}(\tau^{-1}(n))) \tag{2.8}$$

which is equivalent to

$$y(n) \geq P_1^* y(\xi^{-1}(n)) + P_3^* y(\xi(n)) + Q_1^* y(\tau^{-1}(\xi(n))) + Q_2^* y(\tau^{-1}(\tau^{-1}(n))). \tag{2.9}$$

Next, we step by step set

$$n = \xi^{-1}(n), n = \xi(n), n = \tau^{-1}(\xi(n)), n = \tau^{-1}(\tau^{-1}(n))$$

into (2.9). Before substituting, we set

$$Y = y(n), A = y(\xi^{-1}(n)), B = y(\xi(n)), C = y(\tau^{-1}(\xi(n)))$$
$$D = y(\tau^{-1}(\tau^{-1}(n))), E = y(\tau^{-1}(n))$$

Finally, we are let to the following linear inequalities

$$Y \geq P_1^* A + P_3^* B + Q_1^* C + Q_2^* D$$
$$A \geq (P_1^* + P_3^*)Y + Q_1^* E + Q_2^* C$$
$$B \geq P_1^* Y + P_3^* E + Q_1^* D$$

$$C \geq P_1^* E + P_3^* D$$
$$E \geq P_1^* B, D \geq P_1^* C.$$

Eliminating D and E gives

$$Y \geq P_1^* A + P_3^* B + (Q_1^* + P_1^* Q_2^*)C,$$
$$A \geq (P_1^* + P_3^*)Y + P_1^* Q_1^* B + Q_2^* C,$$
$$B \geq P_1^* Y + P_1^* P_3^* B + P_1^* Q_1^* C,$$
$$C \geq (P_1^*)^2 B + P_1^* P_3^* C.$$

Therefore,

$$C \geq \frac{(P_1^*)^2 B}{(1 - P_1^* P_3^*)} = MB$$

which implies

$$B \geq \frac{P_1^*}{1 - P_1^* P_3^* - P_1^* Q_1^* M} Y = NY.$$

Consequently our systems reduces to

$$Y \geq P_1^* A + P_3^* NY + MN(Q_1^* + P_1^* Q_2^*)Y,$$
$$A \geq (P_1^* + P_3^*)Y + P_1^* Q_1^* NY + Q_2^* MNY,$$

which implies that

$$y(n)[1 - P_3^* N - MN(Q_1^* + P_1^* Q_2^*) - P_1^*[P_1^* + P_3^* + P_1^* Q_1^* N + Q_2^* MN] > 0.$$

This contradicts (2.5) and so we conclude that $S_0 = \phi$.

From the proof of Theorem 5, we immediately obtain the following result.

Corollary 6. Assume that there exists a sequence satisfying (2.1) and $\tau^{-1}(n) \geq \sigma(n)$. If

$$C_2 > 1 \text{ or } C_1 C_3 + C_1 d_1 M > 1 \tag{2.10}$$

then $S_0 = \phi$ for (E).

Proof. Proceeding as in the proof of Theorem 5, we are led to (2.8). Clearly $y(n) \geq C_2 y(n)$, which implies $C_1 \leq 1$, a contradiction. Further, from (2.8)

$$Y \geq C_1 A + d_1 C$$
$$\geq C_1 C_3 Y + C_1 d_1 M Y$$

or

$$(1 - C_1 C_3 Y + C_1 d_1 M)Y > 0$$

This contradiction completes the proof.

Next, we use similar method to get criterion for $S_2 = \phi$ for (E). We introduce the sequence $\{\eta(n)\}$ such that

$$\eta(\eta(n)) = \sigma^{-1}(n) \tag{2.11}$$

Lemma 7. Assume that $\{\eta(n)\}$ satisfies (2.11). Then

$$\eta^{-1}(\eta^{-1}(n)) = \sigma(n), \eta(n) = \sigma^{-1}(\eta^{-1}(n)), \eta^{-1}(n) = \sigma(\eta(n)). \tag{2.12}$$

The proof is quite elementary and so we omit it.

Next result concerned with $S_2 = \phi$ for (E). We shall use the notation:

$$Q_1(n) = \sum_{s=\eta(n)}^{n-1} (A(n) - A(s+1))q(s) \geq d_1 > 0$$

$$Q_2(n) = \sum_{s=\sigma^{-1}(n)}^{\eta(n)-1} (A(n) - A(s+1))q(s) \geq d_2 > 0$$

$$Q_3(n) = \sum_{s=\sigma^{-1}(\eta(n))}^{\sigma^{-1}(n)-1} (A(n) - A(s+1))q(s) \geq d_3 > 0$$

and

$$P_1(n) = \sum_{s=\eta(n)}^{n-1} (A(n) - A(s+1))p(s) \geq C_1 > 0$$

$$P_2(n) = \sum_{s=\sigma^{-1}(n)}^{\eta(n)-1} (A(n) - A(s+1))p(s) \geq C_2 > 0$$

Further,

$$Q_i^* = \frac{d_i}{1 - d_2}, i = 1,3, P_i^* = \frac{C_i}{1 - d_2}, i = 1,3$$

and

$$M = \frac{(Q_1^*)^2}{1 - Q_1^* Q_3^*}, N = \frac{Q_1^*}{1 - Q_1^* Q_3^* - Q_1^* P_1^* M}$$

Theorem 8. Assume that there exists a sequence $\{\eta(n)\}$ satisfying (2.11) and $\sigma^{-1}(n) \leq \tau(n)$. If

$$Q_3^* N + MN(P_1^* + Q_1^* P_2^* + Q_1^*(Q_1^* + Q_3^* + Q_1^* P_1^* N + P_2^* MN) > 1, \tag{2.13}$$

then $S_2 = \phi$ for (E).

Proof. Let $\{y(n)\}$ be an eventually positive solution belongs to S_2. Summing the equation (E) twice from n_1 to $n-1$, yields

$$y(n) = \sum_{s=n_1}^{n-1} (A(n) - A(s+1))p(s)y(\tau(s)) + (A(n) - A(s+1))q(s)y(\sigma(s))$$

$$\geq \sum_{s=n_1}^{n-1} (A(n) - A(s+1))p(s)y(\sigma^{-1}(s)) + (A(n) - A(s+1))q(s)y(\sigma(s))$$

Using the auxiliary sequence $\{\eta(n)\}$, we have

$$y(n) \geq \sum_{s=\eta(n)}^{n-1} (A(n) - A(s+1))q(s)y(\sigma(s)) + \sum_{s=\sigma^{-1}(n)}^{\eta(n)-1} (A(n) - A(s+1))q(s)y(\sigma(s))$$

$$+ \sum_{s=\sigma^{-1}(\eta(n))}^{\sigma^{-1}(n)-1} (A(n) - A(s+1))q(s)y(\sigma(s)) + \sum_{s=\eta(n)}^{n-1} (A(n) - A(s+1))p(s)y(\sigma^{-1}(s))$$

$$+ \sum_{s=\sigma^{-1}(n)}^{\eta(n)-1} (A(n) - A(s+1))p(s)y(\sigma^{-1}(s))$$

Since $\{y(n)\}$ is an increasing sequence, we see that

$$y(n) \geq d_1 y(\eta^{-1}(n)) + d_2 y(n) + d_3 y(\eta(n)) + C_1 y(\sigma^{-1}(\eta(n))) + C_2 y(\sigma^{-1}(\sigma^{-1}(n)))$$

that is,

$$y(n) \geq Q_1^* y(\eta^{-1}(n)) + Q_3^* y(\eta(n)) + P_1^* y(\sigma^{-1}(\eta(n))) + P_2^* y(\sigma^{-1}(\sigma^{-1}(n))),$$

which is almost the same inequality as (2.9) in terns of new quantities Q_i and P_i. Hence following the same method of proof as in Theorem 5, we are led to

$$y(n)\{1 - Q_3^* N - MN(P_1^* + Q_1^* P_2^*) - Q_1^*(Q_1^* + Q_3^* + Q_1^* P_1^* N + P_2^* MN)\} \geq 0$$

which contradicts to (2.13) and we conclude that $S_2 = \phi$ for (E).

Corollary 9. Assume that there exists a sequence $\{\eta(n)\}$ satisfying (2.11) and $\sigma^{-1}(n) \leq \tau(n)$. If

$$d_2 > 1 \text{ or } d_1 d_3 + d_3 C_1 M > 1$$

then $S_2 = \phi$ for (E).

Proof. The proof is similar to Corollary 6 and Theorem 8 and so we omit the details.

Combining Theorem 5 and Theorem 8, we have following oscillation criterion.

Theorem 10. Assume that there exists sequence $\{\xi(n)\}$ and $\{\eta(n)\}$ satisfying the conditions (2.1) and (2.11) respectively. If $\tau(\sigma(n)) \leq n \leq \sigma(\tau(n))$ and then conditions (2.5) and (2.13) holds, then equation (E) is oscillatory.

We conclude this paper with an examples which shows the importance of our main results.

Example 11. Consider the second-order difference equation with retarded and advanced argument

$$\Delta^2 y(n) = a\left(\frac{n+1}{n}\right)y(n-2) + b\left(\frac{n+3}{n+2}\right)y(n+2), n \geq 1 \tag{2.14}$$

where $a > 0$ and $b > 0$ are constants.

Here $\tau(n) = n - 2, a(n) = 1, \sigma(n) = n + 2$.

In view of Theorem 1, the class $S_0 = \phi$ for (E) if $a > 1/6$ regardless of value b.

If we choose $\xi(n) = n - 1$, then condition (2.1) holds. By taking $a = 1/8$ and $b = 33$, we have $P_1^* = 1/6, P_3^* = 1/2, Q_1^* = 44, Q_2^* = 88, M = 1/33$ and $N = 6/11$.

The condition (2.5) holds since it talks the form 2.0808 ¿ 1 . Hence the improvement is significant. Moreover, one can reduce the value of needed a by increasing b since the condition (2.5) involve both a and b.

3. Conclusion

In this paper, we present conditions for the oscillation of all solutions of a secondorder difference equation with delay and advanced arguments. The obtained results are new and good contributions to be existing theory of difference equations.

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