THE REVERSE BONDAGE NUMBER OF A GRAPH

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Abstract: The reverse bondage number $b^{-1}(G)$ of a graph G to be the cardinality of a smallest set $E' \subseteq E$ of edges for which $\gamma^{-1}(G-E') > \gamma^{-1}(G)$. Thus, the reverse bondage number of G is the smallest number of edges whose removal will render every minimum inverse dominating set in G a "non inverse dominating set" set in the resultant spanning sub graph.

Keywords: **Domination numbers, Inverse domination numbers, Bondage number**
2000 mathematics subject classification:-05c69,05c70

I. INTRODUCTION:

The graphs considered here are finite, nontrivial, connected, undirected without loops or multiple edges. The study of domination in graphs was begun by Ore and Berge [1]. A set D of vertices in a graph G is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the order of a smallest dominating set in G. In a communication network, let D denote the set of transmitting stations so that every station not belonging to D have a link with at least one station in D. If this set of stations fails, then one has to find another disjoint such set of stations. This leads to define the domatic number. The domatic number d (G) of G is the maximum number of dominating sets in G [2]. Kulli and Sigarkanti consider the problem of selecting such set of transmitting stations so that if it fails and the existence of a disjoint set of transmitting stations should contains minimum number of stations, or if it is required to work on two disjoint set of transmitting stations D_1 (D_2) so that each station not belonging to D_1 (D_2) have a link with at least one station in D_1 (D_2). This leads us to define the inverse domination number. Let D be a minimum dominating set of G. If V-D contains a dominating set say D^1 of G, then D^1 is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the order of a smallest inverse dominating set in G.

This concept was first defined by Kulli and Sigarkanti [2]. Application concerning the vulnerability of the communications network under link failure, was given in [3]. In particular, suppose that some one does not know which sites in the network act as transmitters, but does know that the set of such site corresponds to a minimum dominating set in the related graph. What is the fewest number of communication links that he must sever so that at least one additional transmitter would be received in order that communication with all sites be possible ? With this idea we now introduce inverse bondage number of a graph.

II. SOME EXACT VALUES.

We begin our investigation of the reverse bondage number by computing the value for several well known classes of graphs.

Proposition 1 : The reverse bondage number of the complete graph K_p ($p\geq3$) is $b^{-1}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor$

Proof : If H is a spanning of K_p that is obtained subgraph by removing fewer than $\left\lfloor \frac{p}{2} \right\rfloor$ edges from K_p, then H contains at least two vertices of degree p – 1, Hence $\gamma^{-1}(H)=1$. Thus $b^{-1}(Kp) \geq \left\lfloor \frac{p}{2} \right\rfloor$.

Case (1) : If p is even, the removal of $\left\lfloor \frac{p}{2} \right\rfloor$ independent edges from K_p reduces the degree of each vertex to p – 2 and therefore yields a graph H with inverse domination number $\gamma^{-1}(H)=2$.

Case (2) : If p is odd, the removal of $\frac{p-1}{2}$ independent edges from K_p leaves a graph H having exactly one vertex of degree p – 1 and rest of the vertices having degree p – 2 hence $\gamma^{-1}(H)=2$.

In both cases the graph H resulted by the removal of $\left\lfloor \frac{p}{2} \right\rfloor$ edges from K_p. Thus, $b^{-1}(Kp) = \left\lfloor \frac{p}{2} \right\rfloor$

Next we find the reverse bondage number of a cycle C_p with p vertices as follows.

Proposition 2 : For any cycle C_p other than C_4

$$b^{-1}(C_p) = \begin{cases} 1 & \text{if } p\equiv 0(\text{mod}3) \\ 2 & \text{otherwise} \end{cases}$$

Proof : Due to Kulli and Sigarkanti [58], $\gamma^{-1}(C_p) = \left\lceil \frac{p}{3} \right\rceil$ The removal of an edge from C_p results a path P_p.

Case (1) If $p \equiv 0 \pmod 3$, then again by Kulli and Sigarkanti [58] $\gamma^{-1}(P_p) = \left\lceil \frac{p}{3} \right\rceil + 1$.Hence, $b^{-1}(C_p) = 1$

Case (2) : If $p \neq 0(\text{mod}\,3)$, then by Kulli and Sigarkanti [58], $\gamma^{-1}(P_p) = \left\lceil \frac{p}{3} \right\rceil$ If we remove an edge from path P_p results a graph H having two paths P_{p_1} and P_{p_2} such that either $p_1 \equiv 0 \pmod 3$ or $p_2 \equiv 0 \pmod 3$.

Let $p_1 \equiv 0$ mod(3). Then, $\gamma^{-1}(P_{p_1}) = \left\lceil \frac{p_1}{3} \right\rceil + 1$ Hence , $\gamma^{-1}(P_{p_1}) + \gamma^{-1}(P_{p_2}) > \gamma^{-1}(P_p) = \gamma^{-1}(C_p)$ .Thus, $b^{-1}(C_p) = 2$

In the next result we find reverse bondage number of a path P_p with p vertices.

Proposition 3 : For any path P_p with order $p \geq 5$

$$b^{-1}(P_p) = 1$$

Proof : Let G be a path with $p \geq 5$ vertices by Kulli and Sigarkanti [58].

$$\gamma^{-1}(P_p) = \begin{cases} \left\lceil \frac{p}{3} \right\rceil + 1 & \text{if } p \equiv 0 \pmod 3 \\ \left\lceil \frac{p}{3} \right\rceil & \text{otherwise} \end{cases}$$

Case (1) : If $p \neq 0 \pmod 3$, then remove an edge from P_p results a graph H consisting of two paths P_{p_1} and P_{p_2} such that either p_1 or $p_2 \equiv 0 \pmod 3$. Hence by proposition 2, $\gamma^{-1}(H) > \gamma^{-1}(P_p)$. Hence, $b^{-1}(P_p) = 1$

Case (2) : If $p \equiv 0 \pmod 3$, then remove an edge from P_p results a graph H consisting of two paths P_{p_1} and P_{p_2} such that both $p_1 \equiv 0 \pmod 3$ and $p_2 \equiv 0 \pmod 3$.

Hence, $\gamma^{-1}(P_{p_1}) = \left\lceil \frac{p_1}{3} \right\rceil + 1$

$$\gamma^{-1}(P_{p_2}) = \left\lceil \frac{p_2}{3} \right\rceil + 1$$

Thus, $\gamma^{-1}(P_{p_1}) + \gamma^{-1}(P_{p_2}) = \left\lceil \frac{p_1}{3} \right\rceil + \left\lceil \frac{p_2}{3} \right\rceil + 2 > \left\lceil \frac{p}{3} \right\rceil + 1$,so that $\gamma^{-1}(H) > \gamma^{-1}(P_p)$ Hence, $b^{-1}(P_p) = 1$

Proposition 4 : Let $K_{m,n}$ be a complete bipartite graph other than C_4 with $2 \leq m \leq n$. Then $b^{-1}(K_{m,n}) = m - 1$

Proof : Let $V = V_1 \cup V_2$ be the vertex set of $K_{m,n}$, where $|V_1|=m$ and $|V_2|=n$. Let $v \in V_2$. Then by removing $(m - 1)$ edges incident with v, we obtain a bipartite graph G^1 with exactly one end vertex. Hence

$$\gamma^{-1}(G^1) = 3 > 2 = \gamma^{-1}(G) \quad \text{Thus } b^{-1}(K_{m,n}) = m - 1$$

Proposition 5 : For any wheel W_p, $b^{-1}(W_p) = 2$

Proof : Let $W_p = 1 + C_{p-1}$ with vertex set

$v_o \cup \{v_1, v_2 \ldots, v_p\}$, where v_o is the vertex of degree $p-1$ and degree $v_i=3$, for $i = 1, 2, \ldots, p-1$. Then removal adjacent edges $\{v_1v_2, v_2v_3\}$ from W_p yields a graph H with exactly one end vertex. Hence, $\gamma^{-1}(H) = 1 + \gamma(P_{p-2})$

$$= 1+\left\lceil \frac{p-2}{3} \right\rceil = \left\lceil \frac{p}{3} \right\rceil > \left\lceil \frac{p-1}{3} \right\rceil = \gamma^{-1}(W_p) \text{ Thus } b^{-1}(W_p) = 2$$

GENERAL BOUNDS

In the following proposition we give on upper bound on reciprocal bondage number interms degree of G.

Proposition 6 : If G is a non empty graph then, $b^{-1}(G) \leq \min\{\deg u + \deg v - 2 : u \text{ and } v \text{ are adjacent vertices}\}$.

Proof : Let k denote the right hand side of the inequality above, and let u and v are adjacent vertices of G such that $\deg u + \deg v - 2 = k$. Assume that $b^{-1}(G) > k$. Let E_u denote the set of edges that are incident with a vertex u which have minimum degrees among u and v, then $|E_u| = k$ and $\gamma^{-1}(G-E_u) = \gamma^{-1}(G)$ which is a contradiction since u is a isolated vertex in $G-E_u$ and hence $\gamma^{-1}(G-E_u)$ does not exists. Thus $b^{-1}(G) \leq k = \min\{\deg u + \deg v - 2 : u \text{ and } v \text{ are adjacent}\}$.

Proposition 7 : If $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degrees of a non empty connected graph G, then $b^{-1}(G) \leq \Delta(G) + \delta(G) - 2$.

Proof : Let v be a vertex with minimum degree $\delta(G)$ of a non empty connected graph G. Since G has no isolates v must be adjacent to a vertex u. By proposition 6, $b^{-1}(G) \leq \deg u + \deg v - 2$.

$$\leq \deg u + \delta(G) - 2 \text{ Since } \deg u \leq \Delta(G), \text{ hence}$$
$$b^{-1}(G) \leq \Delta(G) + \delta(G) - 2$$

Proposition 8 : for any graph G, $\gamma(G) + \gamma^{-1}(G) = p$ if any only if $b^{-1}(G)=0$

Proof : suppose $\gamma(G) + \gamma^{-1}(G) = p$ then obviously $b^{-1}(G)=0$. Conversely, let $b^{-1}(G) = 0$. Due to Kulli and Sigarkanti [58] for any graph G $\gamma(G)+\gamma^{-1}(G) \leq p$. Suppose $\gamma(G) + \gamma^{-1}(G) < p$.

Then there is a vertex $u \in V(G)$ such that $u \notin D \cup D^1$ where D is a minimal dominating set in G and D^1 is a minimal inverse dominating set in G, but is adjacent to vertices of D and D^1. Let E^1 denote the set of edges incident with u from the vertices of D^1. Then $\gamma^{-1}(G - E^1) > \gamma^{-1}(G)$ which is contradiction, since $b^{-1}(G)=0$.

Hence $\gamma(G) + \gamma^{-1}(G) = p$.

In the following theorem, we give Nordhaus-Gaddum type results.

Proposition 9 : Let G be a graph for which both $b^{-1}(G)$ and $b^{-1}(\overline{G})$ exits. Then

(i) $b^{-1}(G) + b^{-1}(\overline{G}) \leq 2(p-3)$

ii) $b^{-1}(G) \,.\, b^{-1}(\overline{G}) \leq (2p - \Delta - 2\delta - 2)\,\Delta + (2p - \delta - 2\Delta - 2)\delta + 8$

Proof : By proposition 7,

we have $b^{-1}(G) \leq \delta(G) + \Delta(G) - 2$ $b^{-1}(\overline{G}) \leq \delta(\overline{G}) + \Delta(\overline{G}) - 2$

Hence $b^{-1}(G) + b^{-1}(\overline{G}) \leq (\delta(G) + \Delta(G) - 2) + (\delta(\overline{G}) + \Delta(\overline{G}) - 2)$.

$$= (\delta(G) + \Delta(G) - 2) + (p - 1 - \Delta(G) + p - 1 - \delta(G) - 2).$$
$$= 2(p-3).$$

Also

$$b^{-1}(G) \,.\, b^{-1}(\overline{G}) \leq (\delta(G) + \Delta(G) - 2) \,.\, (\delta(\overline{G}) + (\overline{G}) - 2).$$
$$= (\delta(G) + \Delta(G) - 2) \,.\, (p - 1 - \Delta(G) + p - 1 - \delta(G) - 2).$$
$$= (\delta(G) + \Delta(G) - 2) \,.\, (2p - \Delta(G) - \delta(G) - 4).$$
$$= (2p - \Delta - 2\delta - 2)\,\Delta + (2p - \delta - 2\Delta - 2)\delta + 8.$$

ACKNOWLEDGEMENT:

The authors wish to thank Rajath Giri D T for his help and valuable suggestions in the preparation of this paper

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