# A SEVENTH ORDER TRANSFORMATION METHOD FOR MULTIPLE ROOTS AND ITS BASINS OF ATTRACTION

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**Abstract** In this contribution, a novel seventh-order transformation method is proposed and analyzed for finding multiple roots of nonlinear equations, when the multiplicity of the root is not known explicitly. The proposed method does not require the evaluation of second derivative. The basins of attraction of the proposed method are also presented in comparison to existing transformation methods in the literature.

**Keywords:** Nonlinear equations· Iterative method· Multiple root· Order of convergence· Basins of Attraction.

**Mathematics Subject Classifications (2010):**

65B99, 65H05.

## 1 Introduction

In this study, we apply iterative methods to find a multiple root $\alpha$ of multiplicity $m > 1$, i.e. $f^{(j)}(\alpha) = 0$, $j = 0, 1, ....m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$, where $f(x)$ be the continuously differentiable real or complex function. Modified Newton method [1] is an important and basic method for finding multiple roots

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \tag{1}$$

which converges quadratically and requires the knowledge of multiplicity $m$ of root $\alpha$.
In order to improve the order of convergence of (1), several higher-order methods have been proposed in the literature with known multiplicity $m$, for example, [2–28]. On the other hand, if multiplicity $m$ is not known explicitly, Traub [29] suggested a simple transformation:

$$F(x) = \begin{cases} \dfrac{f(x)}{f'(x)} & \text{if } f(x) \neq 0\ , \\ 0 & \text{if } f(x) = 0, \end{cases} \tag{2}$$

to find a multiple root of $f(x) = 0$, thereby reducing the task of finding a multiple root to that of solving a simple root of the transformed equation $F(x) = 0$. Thus any iterative method can be used to preserve the original order of convergence. However, with this transformation, we get second order transformed Newton method given by

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}, \tag{3}$$

which requires the use of $f'(x)$ and $f''(x)$. In order to avoid the calculations of these derivatives, King [30] proposed the secant method, with unknown multiplicity for finding multiple roots of nonlinear equation, which used another transformation:

$$F(x) = \frac{-f^2(x)}{f(x - f(x)) - f(x)}. \tag{4}$$

The secant method thus obtained has order of convergence 1.618.
Using the same transformation (4), Iyengar and Jain [31] developed two iterative methods of order three and four for finding multiple roots of nonlinear equations. The third order method is given as:

$$x_{k+1} = x_k - l_1 - l_2, \tag{5}$$

where

$$\begin{aligned} l_1 &= \frac{F(x_k)}{G(x_k)}, \quad l_2 = \frac{F(x_k - l_1)}{G(x_k)}, \\ G(x_k) &= \frac{F(x_k + \beta F(x_k)) - F(x_k)}{\beta F(x_k)}. \end{aligned} \tag{6}$$

and fourth order method is expressed as:

$$x_{k+1} = x_k - l_1 - l_2 - l_3, \tag{7}$$

where $l_1$ and $l_2$ are as defined in (6) and $l_3 = \dfrac{F(x_k - l_1 - l_2)}{G(x_k)}$.

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With the same transformation (4), Wu and Fu [32] developed a quadratically convergent iterative method for multiple roots given as:

$$x_{k+1} = x_k - \frac{F^2(x_k)}{p.F^2(x_k) + F(x_k) - F(x_k - F(x_k))}, \quad (8)$$

where $p \in \mathbb{R}, |p| < \infty$.
Wu et. al. [33] suggested another transformation:

$$F(x) = \begin{cases} \dfrac{\text{sign}(f(x))f(x)|f(x)|^{1/m}}{\text{sign}(f(x+\text{sign}(f(x))|f(x)|^{1/m}) - f(x))f(x)|f(x)|^{1/m} + f(x+\text{sign}(f(x))|f(x)|^{1/m}) - f(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0, \end{cases}$$

and proposed a quadratically convergent method by applying this transformation to modified Steffensen's method [34, 35].

Parida and Gupta [36] proposed another transformation:

$$F(x) = \begin{cases} \dfrac{f^2(x)}{\text{sign}(f(x+f(x)) - f(x))f^2(x) + f(x+f(x)) - f(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0, \end{cases}$$

and obtained a quadratically convergent iterative method given as:

$$x_{k+1} = x_k - \frac{F^2(x_k)}{p.F^2(x_k) + F(x_k) - F(x_k - F(x_k))}, \quad (9)$$

where the parameter $p$ should be chosen such that the denominator is largest in magnitude.
Yun [37] also suggested another transformation for finding multiple root $\alpha \in (a, b)$ of $f(x) = 0$ given as:

$$F(x) = \frac{\epsilon f^2(x)}{f(x + \epsilon f(x)) - f(x)},$$

where $\epsilon$ is such that $\max\limits_{a \le x \le b} |\epsilon f(x)| = \delta$. Using this transformation, Yun proposed a quadratically convergent iterative method as follows:

$$x_{k+1} = x_k - \frac{2(x_k - x_{k-1})F(x_k)}{F(2x_k - x_{k-1}) - F(x_{k-1})}. \quad (10)$$

In recent years, various higher order transformation methods have been developed and analyzed. Li et. al. [38] used the transformation (2) and proposed a fifth-order iterative method for multiple roots of the nonlinear equation $f(x) = 0$, which is given as,

$$\begin{cases} w_k = x_k - \dfrac{F(x_k)}{g(x_k)}, \\ z_k = w_k - \dfrac{F(w_k)F(x_k)}{F(x_k + F(x_k)) - F(x_k)}, \\ x_{k+1} = z_k - \dfrac{F(z_k)}{F[z_k, w_k] + F[z_k, x_k, x_k](z_k - w_k)}, \end{cases} \quad (11)$$

where $F[.,.]$ and $F[.,.,.]$ are divided differences of $F$ of order one and two respectively and

$$g(x_k) = \frac{F(x_k + F(x_k)) - F(x_k)}{F(x_k)}, \quad (12)$$

More recently, Sharma et al. [39] developed and analyzed a transformation method of sixth order for finding multiple roots of nonlinear equations with unknown multiplicity $m$ :

$$\begin{cases} w_k = x_k - \dfrac{F(x_k)}{g(x_k)}, \\ z_k = w_k - \dfrac{F(w_k)}{g(x_k)}, \\ x_{k+1} = z_k - \dfrac{F(z_k)}{G(x_k, w_k, z_k)}, \end{cases} \quad (13)$$

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where $F(x_k)$, $g(x_k)$ are given by (2) and (12) respectively
and

$$G(x_k, w_k, z_k) = \frac{F[x_k, z_k]F[w_k, z_k]}{F[x_k, w_k]} \tag{14}$$

Inspired by the ongoing work in this direction, we here propose a novel modification of Newton method based on the transformation (2). The proposed method is composed of three steps per iteration and is of seventh order convergence, without requiring the use of second derivative.

The paper is organized as follows. In section 2, a seventh-order method for multiple roots, with unknown multiplicity $m$ is proposed and its convergence behavior is discussed. In section 3, a comparison of basins of attraction is provided to illustrate the convergence behavior of the proposed schemes in complex plane. Concluding remarks are given in section 4.

## 2 The Method and its Convergence analysis

We here use the transformation (2) and consider the following iteration scheme:

$$\begin{cases} y_k = x_k - \dfrac{F(x_k)}{F'(x_k)}, \\ z_k = y_k - \dfrac{F(y_k)}{F'(y_k)}, \\ x_{k+1} = z_k - \dfrac{F(z_k)}{F'(z_k)}, \end{cases} \tag{15}$$

where $F(x)$ is given in (2).

In order to avoid evaluation of first derivatives, we approximate function $F'(x)$ and $F'(y(x))$ by using Eq. (12) and further approximate $F'(y(x))$ and $F'(z(x))$ by using Forward difference operator and Lagrange Interpolation respectively as given below:

$$F'(w_k) \approx \frac{F(w_k + F(w_k)) - F(w_k)}{F(w_k)} = H(w_k). \tag{16}$$

$$F'(z_k) \approx F[x_k, z_k] + F[y_k, z_k] - F[x_k, y_k] = \phi(x_k, y_k, z_k), \tag{17}$$

where $F[.,.]$ denotes the first order divided difference. (See [40] for the detailed discussion of 17). Replacing the approximations from (12), (16) and (17) in (15), the proposed scheme in the final form is given as:

$$\begin{cases} y_k = x_k - \dfrac{F(x_k)}{g(x_k)}, \\ z_k = y_k - \dfrac{F(y_k)}{H(y_k)}, \\ x_{k+1} = z_k - \dfrac{F(z_k)}{\phi(x_k, y_k, z_k)}. \end{cases} \tag{18}$$

The mathematical proof for the order of convergence of this scheme (18) is given in following theorem.

**Theorem 1.** *Let $F \in C^2(I)$ $(I \subseteq \mathbb{R} \to \mathbb{R})$ has a simple root $r \in I$, where $I$ is an open interval. If the initial point $x_0$ is sufficiently close to $r$, then the iterative method defined by (18) has seventh order convergence.*

*Proof.* Since $\alpha$ is a multiple root of $f(x) = 0$ with multiplicity $m$, so we can write $f(x)$ as

$$f(x) = (x - \alpha)^m h(x), \tag{19}$$

where $h(x)$ is a continuous function with $h(\alpha) \neq 0$. According to (19), we have

$$f'(x) = m(x - \alpha)^{m-1} h(x) + (x - \alpha)^m h'(x). \tag{20}$$

Dividing (19) by (20), we get

$$F(x) = \frac{f(x)}{f'(x)} = \frac{(x - \alpha)h(x)}{mh(x) + (x - \alpha)h'(x)}. \tag{21}$$

Consequently, the problem of finding multiple roots of $f(x) = 0$ can be reduced to equivalent problem of finding simple root of $F(x) = 0$.

Let $e_k = x_k - \alpha$ be the error in the iterate $x_k$. Using Taylor series expansion, we get

$$\begin{aligned} h(x_k) &= h(\alpha)[1 + A_1 e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + A_5 e_k^5 \\ &\quad + A_6 e_k^6 + A_7 e_k^7 + O(e_k^8)], \end{aligned} \tag{22}$$

$$\begin{aligned} h'(x_k) &= h(\alpha)[A_1 + 2A_2 e_k + 3A_3 e_k^2 + 4A_4 e_k^3 + 5A_5 e_k^4 \\ &\quad + 6A_6 e_k^5 + 7A_7 e_k^6 + O(e_k^7)], \end{aligned} \tag{23}$$

where

$$A_k = \frac{h^{(k)}(\alpha)}{k! h(\alpha)}, \ k = 1, 2, \ldots \tag{24}$$

Using (21), (22) and (23) and simplifying, we get

$$\begin{aligned} F(x_k) &= B_1 e_k + B_2 e_k^2 + B_3 e_k^3 + B_4 e_k^4 + B_5 e_k^5 + B_6 e_k^6 \\ &\quad + O(e_k^7), \end{aligned} \tag{25}$$

where

$$\begin{aligned} B_1 &= \frac{1}{m}, \quad B_2 = -\frac{A_1}{m^2}, B_3 = \frac{(m+1)A_1^2 - 2mA_2}{m^3}, \\ B_4 &= \frac{-(m+1)^2 A_1^3 + (3m^2 + 4m)A_1 A_2 - 3m^2 A_3}{m^4}, \\ B_5 &= \frac{1}{m^5}\Big[(m+1)^3 A_1^4 + (-4m^3 - 10m^2 - 6m)A_1^2 A_2 \\ &\quad + (2m^3 + 4m^2)A_2^2 + (4m^3 + 6m^2)A_1 A_3 - 4m^3 A_4\Big], \\ B_6 &= \frac{1}{m^6}\Big[ - (m+1)^4 A_1^5 + (m+1)^2(8 + 5m)A_1^3 A_2 \\ &\quad -(5m^4 + 14m^3 + 9m^2)A_1^2 A_3 - (5m^4 + 16m^3 \\ &\quad + 12m^2)A_1 A_2^2 + (5m^4 + 8m^3)A_1 A_4 \\ &\quad + (5m^4 + 12m^3)A_2 A_3 - 5m^4 A_5\Big]. \end{aligned}$$

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Substituting (25) in (12) and using Taylor series expansion of $F(x_k)$ and $F(x_k+F(x_k))$ , we get

$$\begin{aligned} g(x_k) &= \frac{1}{m}+C_1e_k+C_2e_k^2+C_3e_k^3+C_4e_k^4+C_5e_k^5 \\ &\quad +C_6e_k^6+O(e_k^7), \end{aligned} \tag{26}$$

where

$$\begin{aligned}
C_1 &= -\frac{(2m+1)A_1}{m^3}, \\
C_2 &= \frac{(3m^3+6m^2+5m+1)A_1^2-2m(3m^2+3m+1)A_2}{m^5}, \\
C_3 &= \frac{1}{m^7}\Big[(2m+1)(2m^4+6m^3+9m^2+6m+1)A_1^3 \\
&\quad -m(6m^3+14m^2+15m+4)A_1A_2+3m^2(2m^2 \\
&\quad +2m+1)A_3\Big], \\
C_4 &= \frac{1}{m^9}\Big[(5m^7+25m^6+65m^5+100m^4+90m^3 \\
&\quad +45m^2+11m+1)A_1^4-m(20m^6+90m^5 \\
&\quad +203m^4+252m^3+165m^2+52m+6)A_1^2A_2 \\
&\quad +m^2(20m^5+70m^4+121m^3+104m^2+43m \\
&\quad +6)A_1A_3+2m^2(5m^5+20m^4+36m^3+29m^2 \\
&\quad +11m+2)A_2^2-4m^3(5m^4+10m^3+10m^2 \\
&\quad +5m+1)A_4\Big], \\
C_5 &= \frac{1}{m^{11}}\Big[-(m+1)^2(6m^7+27m^6+76m^5+120m^4 \\
&\quad +110m^3+53m^2+12m+1)A_1^5+m(30m^8 \\
&\quad +183m^7+585m^6+1137m^5+1365m^4+1000m^3 \\
&\quad +424m^2+93m+8)A_1^3A_2-m^2(30m^7+159m^6 \\
&\quad +435m^5+693m^4+651m^3+348m^2+92m+9)A_1^2A_3 \\
&\quad -m^2(30m^7+171m^6+488m^5+791m^4+726m^3 \\
&\quad +373m^2+104m+12)A_1A_2^2+m^3(30m^6+147m^5 \\
&\quad +334m^4+375m^3+228m^2+77m+12)A_2A_3 \\
&\quad +m^3(30m^6+123m^5+264m^4+315m^3+210m^2 \\
&\quad +69m+8)A_1A_4-5m^4(6m^5+15m^4+20m^3 \\
&\quad +15m^2+6m+1)A_5\Big], \\
C_6 &= \frac{1}{m^{13}}\Big[(m+1)^2(7m^9+42m^8+154m^7+350m^6 \\
&\quad +511m^5+476m^4+272m^3+89m^2+15m+1)A_1^6 \\
&\quad -m(42m^{10}+322m^9+1328m^8+3498m^7+6115m^6 \\
&\quad +7186m^5+5627m^4+2846m^3+875m^2+146m \\
&\quad +10)A_1^4A_2+m^2(42m^9+294m^8+1092m^7+2526m^6 \\
&\quad +3775m^5+3676m^4+2273m^3+832m^2+159m \\
&\quad +12)A_1^3A_3+m^2(63m^9+462m^8+1776m^7+4212m^6 \\
&\quad +6342m^5+6112m^4+3717m^3+1377m^2 \\
&\quad +282m+24)A_1^2A_2^2-m^3(84m^8+560m^7+1888m^6 \\
&\quad +3728m^5+4447m^4+3248m^3+1437m^2+356m \\
&\quad +36)A_1A_2A_3-m^3(42m^8+252m^7+804m^6 \\
&\quad +1560m^5+1909m^4+1456m^3+641m^2+142m \\
&\quad +12)A_1^2A_4-2m^3(7m^8+49m^7+173m^6+355m^5+423m^4 \\
&\quad +299m^3+129m^2+32m+4)A_2^3 \\
&\quad +m^4(42m^7+196m^6+500m^5+760m^4+701m^3 \\
&\quad +372m^2+101m+10)A_1A_5+2m^4(21m^7+119m^6 \\
&\quad +325m^5+473m^4+403m^3+205m^2+59m \\
&\quad +8)A_2A_4+3m^4(7m^7+42m^6+116m^5+164m^4 \\
&\quad +135m^3+70m^2+22m+3)A_3^2-6m^5(7m^6 \\
&\quad +21m^5+35m^4+35m^3+21m^2+7m+1)A_6\Big].
\end{aligned}$$

Using (25) and (26) in first substep of (18), we obtain

$$\tilde{e_k}=w_k-\alpha=D_2e_k^2+D_3e_k^3+D_4e_k^4+D_5e_k^5+D_6e_k^6+O(e_k^7), \tag{27}$$

where

$$\begin{aligned}
D_2 &= -\frac{(m+1)A_1}{m^2}, \\
D_3 &= \frac{(2m^2+3m+2)A_1^2-(4m^2+6m+2)A_2}{m^3}, \\
D_4 &= \frac{1}{m^4}\big(-(3m^3+5m^2+6m+3)A_1^3+(9m^3+16m^2 \\
&\quad +16m+5)A_1A_2-(9m^3+18m^2+12m+3)A_3\big), \\
D_5 &= \frac{2}{m^5}\Big[(2m^4+3m^3+5m^2+5m+2)A_1^4 \\
&\quad -(8m^4+13m^3+19m^2+15m+4)A_1^2A_2 \\
&\quad +(4m^4+6m^3+6m^2+m-1)A_2^2+(8m^4 \\
&\quad +17m^3+23m^2+16m+5)A_1A_3 \\
&\quad -(8m^4+20m^3+20m^2+10m+2)A_4\Big], \\
D_6 &= \frac{1}{m^6}\Big[-(5m^5+5m^4+11m^3+18m^2+15m+5)A_1^5 \\
&\quad +(25m^5+28m^4+52m^3+71m^2 \\
&\quad +46m+10)A_1^3A_2 \\
&\quad -(25m^5+27m^4+36m^3+29m^2-2m-9)A_1A_2^2 \\
&\quad -(25m^5+42m^4+72m^3+85m^2 \\
&\quad +58m+20)A_1^2A_3+(25m^5+63m^4+108m^3+115m^2 \\
&\quad +70m+17)A_1A_4+(25m^5+33m^4+28m^3 \\
&\quad -3m^2-18m-7)A_2A_3-(25m^5+75m^4+100m^3 \\
&\quad +75m^2+30m+5)A_5\Big].
\end{aligned}$$

Further in (21), using (27) and the Taylor series expansion of $h(w_k)$ and $h'(w_k)$ about $\alpha$, we get

$$F(w_k)=H_2e_k^2+H_3e_k^3+H_4e_k^4+H_5e_k^5+H_6e_k^6+O(e_k^7), \tag{28}$$

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where

$$H_2 = \frac{-(m+1)A_1}{m^3},$$
$$H_3 = \frac{(2m^2+3m+2)A_1^2 - 2(2m^2+3m+1)A_2}{m^4},$$
$$H_4 = \frac{1}{m^6}\Big[-(3m^4+5m^3+7m^2+5m+1)A_1^3 + m(9m^3+16m^2+16m+5)A_1A_2 - 3m(3m^3+6m^2+4m+1)A_3\Big],$$
$$H_5 = \frac{1}{m^7}\Big[2(2m^5+3m^4+7m^3+10m^2+7m+2)A_1^4 - 2(8m^5+13m^4+23m^3+25m^2+12m+2)A_1^2A_2 + 2m(8m^4+17m^3+23m^2+16m+5)A_1A_3 + 2m(4m^4+6m^3+6m^2+m-1)A_2^2 - 4m(4m^4+10m^3+10m^2+5m+1)A_4\Big],$$
$$H_6 = \frac{1}{m^9}\Big[-(5m^7+5m^6+21m^5+47m^4+58m^3+41m^2+14m+1)A_1^5 - m(25m^6+27m^5+52m^4+77m^3+50m^2+15m+4)A_1A_2^2 + m(25m^6+28m^5+86m^4+171m^3+176m^2+94m+20)A_1^3A_2 - m(25m^6+42m^5+90m^4+139m^3+118m^2+50m+6)A_1^2A_3 + m^2(25m^5+33m^4+28m^3-3m^2-18m-7)A_2A_3 + m^2(25m^5+63m^4+108m^3+115m^2+70m+17)A_1A_4 - 5m^2(5m^5+15m^4+20m^3+15m^2+6m+1)A_5\Big].$$

Now, using (27) and (28) in (16) and simplifying, we get

$$H(w_k) = \frac{F(w_k+F(w_k)) - F(w_k)}{F(w_k)} = K_4e_k^4 + K_5e_k^5 + K_6e_k^6 + O(e_k^7), \quad (29)$$

where

$$K_4 = -\frac{{A_1}^3(m+1)^3}{m^7},$$
$$K_5 = \frac{1}{m^8}\big(2{A_1}^2(m+1)^2\big({A_1}^2\big(2m^2+3m+2\big) - 2A_2\big(2m^2+3m+1\big)\big)\big)$$

and

$$K_6 = -\frac{1}{m^{10}}\Big[A_1(m+1)\Big(\Big(10m^5+30m^4+46m^3+40m^2+17m+2\Big){A_1}^4 - 2A_2\Big(17m^5+51m^4+69m^3+48m^2+14m+1\Big){A_1}^2 + 6A_3m(m+1)^2\Big(3m^2+3m+1\Big)A_1 + 4{A_2}^2m\Big(2m^2+3m+1\Big)^2\Big)\Big].$$

Invocation of (27), (28) and (29) in second substep of (18) leads to

$$\hat{e_k} = z_k - \alpha = L_4e_k^4 + L_5e_k^5 + L_6e_k^6 + O(e_k^7), \quad (30)$$

where

$$L_4 = -\frac{{A_1}^3(m+1)^3}{m^6},$$
$$L_5 = \frac{1}{m^7}\Big(2A_1^2(m+1)^2\big(A_1^2\big(2m^2+3m+2\big) - 2A_2\big(2m^2+3m+1\big)\big)\Big)$$

and

$$L_6 = -\frac{1}{m^9}A_1(m+1)\Big[\Big(10m^5+30m^4+46m^3+40m^2+17m+2\Big)A_1^4 - 2A_2\Big(17m^5+51m^4+69m^3+48m^2+14m+1\Big)A_1^2 + 6A_3m(m+1)^2\Big(3m^2+3m+1\Big)A_1 + 4A_2^2m\Big(2m^2+3m+1\Big)^2\Big].$$

Again, using in (21), the Taylor series expansion of $h(z_k)$ and $h'(z_k)$ about $\alpha$, we obtain

$$F(z_k) = M_4e_k^4 + M_5e_k^5 + M_6e_k^6 + O(e_k^7), \quad (31)$$

where

$$M_4 = -\frac{{A_1}^3(m+1)^3}{m^7},$$
$$M_5 = \frac{1}{m^8}\big(2{A_1}^2(m+1)^2\big({A_1}^2\big(2m^2+3m+2\big) - 2A_2\big(2m^2+3m+1\big)\big)\big)$$

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and

$$M_6 = -\frac{1}{m^{10}}\Big[A_1(m+1)\Big(\big(10m^5+30m^4+46m^3 +40m^2+17m+2\big){A_1}^4-2A_2\big(17m^5 +51m^4+69m^3 +48m^2+14m+1\big){A_1}^2 +6A_3m(m+1)^2\big(3m^2+3m+1\big)A_1 +4{A_2}^2m\big(2m^2+3m+1\big)^2\Big)\Big].$$

Employing (29), (30) and (31) in third substep of (18), and applying computer software like MATHEMATICA [41], we have

$$e_{k+1} = -\frac{A_1^4(m+1)^4\left(A_1^2(m+1)-2A_2m\right)e_k^7}{m^{10}} + {e_k}^8.$$

This completes the proof.

We further consider finding the multiplicity of the root $\alpha$ in the iterative method. If $x_k$ is the $k$-th iteration computed by an iterative method applied to $F(x)$, then from (21), we get

$$F(x_k) \approx \frac{(x_k-\alpha)h(x_k)}{mh(x_k)+(x_k-\alpha)h'(x_k)} = \frac{e_kh(x_k)}{mh(x_k)+e_kh'(x_k)}.$$

Since $e_k$ is small, we get $F(x_k) \approx \frac{e_k}{m}$. Similarly, $F(x_{k+1}) \approx \frac{e_{k+1}}{m}$. Also $e_{k+1}-e_k = x_{k+1}-x_k$. Hence, we have,

$$m \approx \frac{x_{k+1}-x_k}{F(x_{k+1})-F(x_k)},$$

which is approximately the reciprocal of divided difference of $F$ for successive iterates $x_k$ and $x_{k+1}$. (see [30, 36]). □

# 3 Finding the basins

In this section, we present the comparison of iterative schemes in the complex plane using basins of attraction. Cayley [42] was the first who considered Newton method for the roots of polynomial with iterations over the complex numbers. The performance of the presented seventh order transformation method denoted by $M_7$ Eq. (18) is compared with some of the existing transformation methods viz. second order transformed Newton method ($NM_2$), Eq. (3), Wu and Fu's second order method designated as $WFM_2$, Eq.(8) for $p = 1$, Iyengar and Jain's third order and fourth order methods respectively denoted by $IJM_3$, Eq.(5) and $IJM_4$, Eq.(7) for $\beta = -7/10$. The fifth order method by Li et al., Eq.(11) and the sixth order method in [39] given by Eq. (13) are also considered for comparison written as $LM_5$ and $M_6$ respectively.

To generate the basins, we use MATHEMATICA [41]. We assign the light to dark colors based on the number of iterations in which the considered initial point $z_0$ converges to a root and we can mark this point with a color associated to this root. We mark with black, the points $z_0$ for which the corresponding iterative method starting in $z_0$ does not reach any root of the polynomial, with tolerance $\epsilon = 10^{-3}$ in a maximum of 40 iterations (see [43–46]).

We have used the considered transformation methods for the test functions as listed in Table 1.

It is noteworthy that transformed Newton method ($NM_2$) is not considered in the competition as it involves second order derivative also. This is only used as a standard measure.

Table 1: Test functions

| $f(z)$ | | | all roots |
|---|---|---|---|
| $f_1(z)$ | $=$ | $(z^2-2z)^2$ | $0, 2$ |
| $f_2(z)$ | $=$ | $(z^5+2z^4+2z^3+10z^2+25z)^2$ | $-2, 0, 1, \pm 2i$ |
| $f_3(z)$ | $=$ | $(z^3-1)^4$ | $-0.5 \pm 0.866025i,\ 1$ |
| $f_4(z)$ | $=$ | $(z^5-z)^4$ | $0,\ \pm 1,\ \pm i$ |

It can be clearly observed from the figures (1-4), that the proposed method $M_7$, Eq. (18) behaves well in almost all the examples.

# 4 Conclusions

In this work, we have proposed a novel and efficient transformation method of seventh-order for finding multiple roots of nonlinear equation $f(x) = 0$, when multiplicity $m$ is not known explicitly. The advantage of the proposed method is that it does not use second order derivative. The presented numerical experiments and basins of attraction show the good performance of the proposed method as compared to other transformation methods in the literature.

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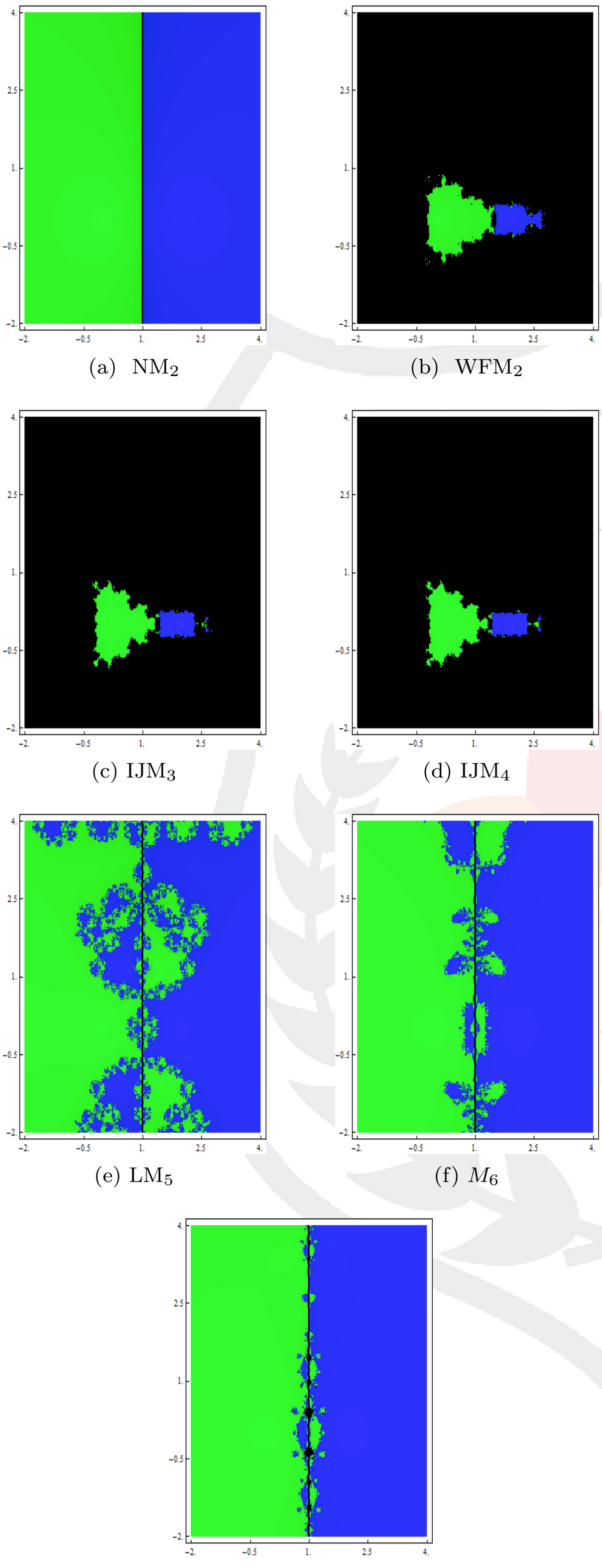
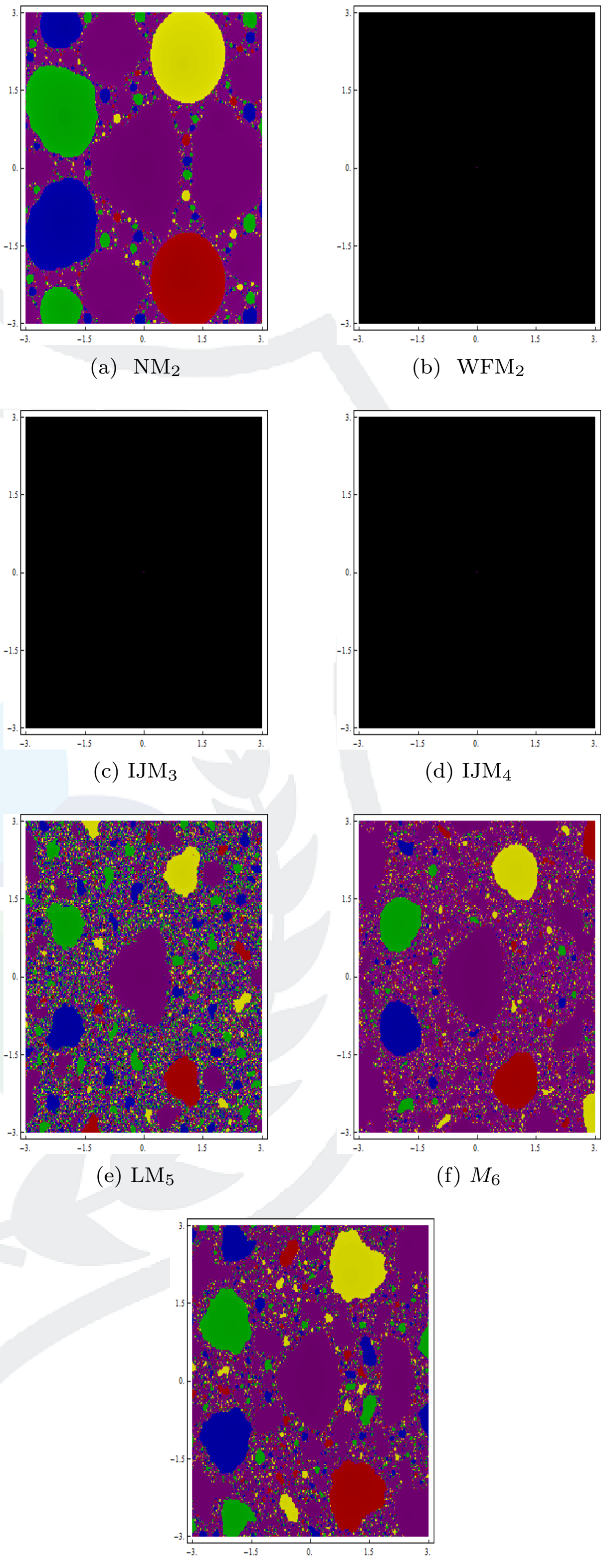


(a) $NM_2$ (b) $WFM_2$ (c) $IJM_3$ (d) $IJM_4$ (e) $LM_5$ (f) $M_6$ (g) $M_7$

Figure 1: Basins of attraction for $f(z) = (z^2 - 2z)^2$, $z \in D$ for various methods.



(a) $NM_2$ (b) $WFM_2$ (c) $IJM_3$ (d) $IJM_4$ (e) $LM_5$ (f) $M_6$ (g) $M_7$

Figure 2: Basins of attraction for $f(z) = (z^5 + 2z^4 + 2z^3 + 10z^2 + 25z)^2$, $z \in D$ for various methods.

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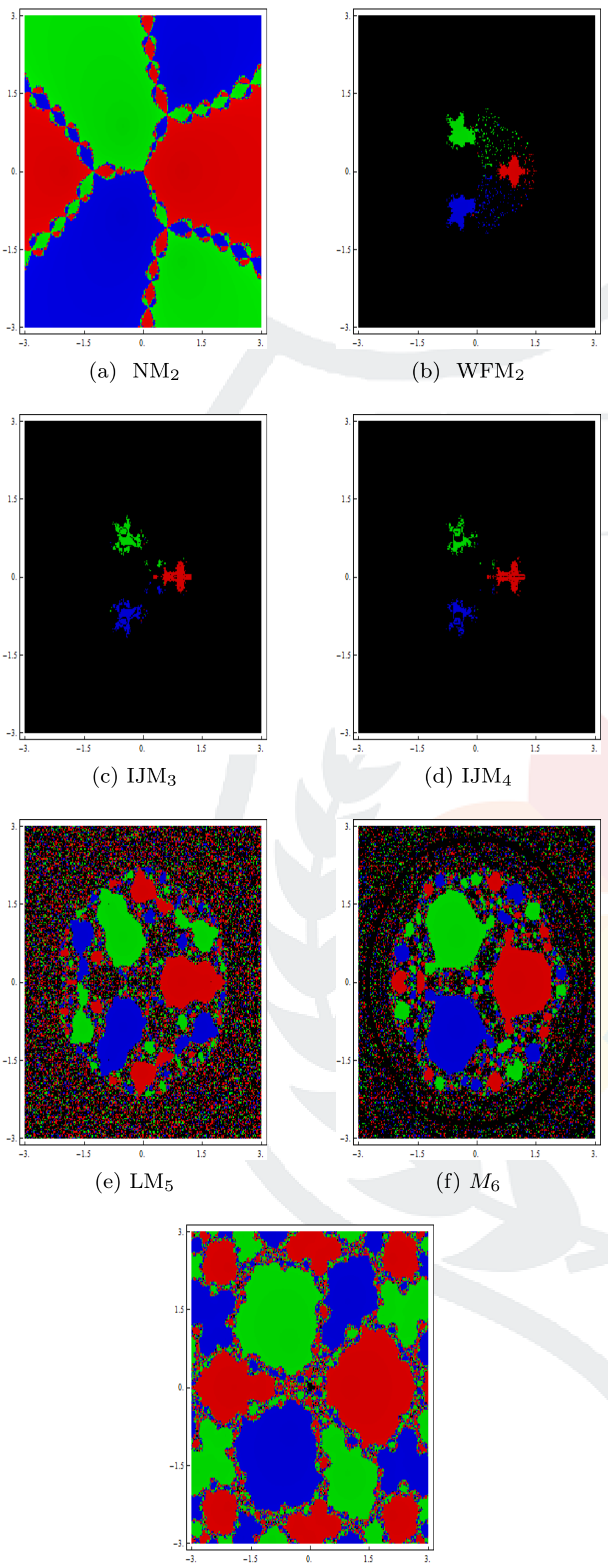

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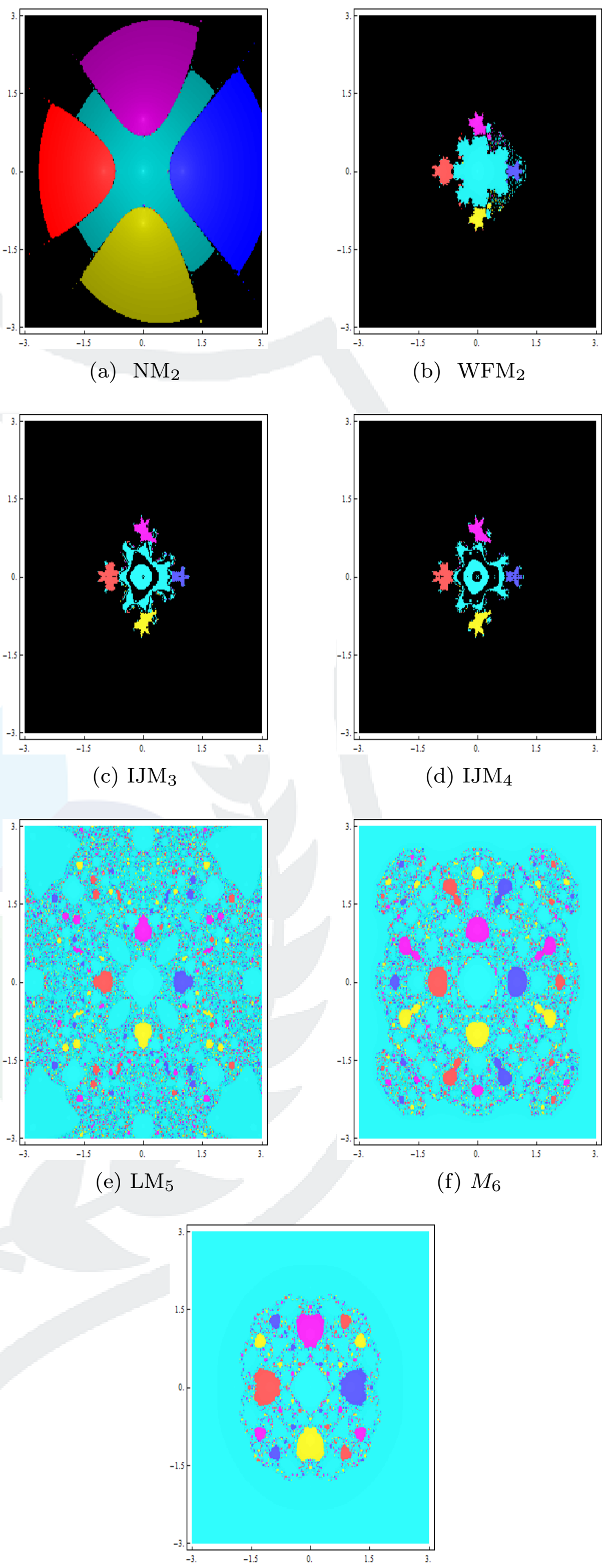
Figure 3: Basins of attraction for $f(z) = (z^3 - 1)^4$, $z \in D$ for various methods.



(a) $NM_2$ (b) $WFM_2$ (c) $IJM_3$ (d) $IJM_4$ (e) $LM_5$ (f) $M_6$ (g) $M_7$

Figure 4: Basins of attraction for $f(z) = (z^5 - z)^4$, $z \in D$ for various methods.

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