Pendant Total Domination Polynomial Of Some Graphs

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Abstract : Let G = (V, E) be a simple graph of order n. The total dominating set S of V that every vertex of V (including those in the set itself) is adjacent to some vertices of S. A total dominating set S of G is called a pendant total dominating set if the induced sub graph <S> contains at least one pendant vertex. In this paper, we introduce a new type of graph polynomial called pendant total domination polynomial. We obtain the pendant total domination polynomial of some standard graphs.

Key Words: Total dominating sets, Pendant total dominating set, Total domination polynomial, Pendant total domination polynomial.

I. Introduction

Throughout this paper, we will consider only simple graphs. Let G = (V,E) be a simple graph. For $S \subseteq V(G)$, we use <S> for the sub graph induced by S. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is the set $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$ .A set of vertices S in a graph G is said to be a dominating set if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S.

A set of vertices S in a graph G is said to be a total dominating set if every vertex $v \in V(G)$ (including those in the set itself) is adjacent to an element of S.A total dominating set S of G is called pendant total dominating set if the induced sub graph <S> contains at least one pendant vertex. The minimum cardinality of a pendant total dominating set of G is called the pendant total domination number and is denoted by $\gamma_{pet}(G)$. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. For a detailed treatment of the domination and domination polynomial of a graph, the reader may referred to [2,1].We introduced the pendant total domination polynomial of G, and obtain pendant total domination polynomial for some standard graphs.The join of G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph such that $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1), v \in V(G_2)\}$ and $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$. The star graph $K_{1,n}$ is a graph of order $n+1$ obtained by joining the two graphs $G_1 \cong K_1$ and $G_2 \cong \overline{K_n}$ .A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} \vee P_n$, where $\overline{K_m}$ is the empty graph on m nodes and P_n is the path graph on n nodes. An m - gonal n - cone graph, also called the n – point suspension of C_m , is defined by the graph join $C_m \vee \overline{K_n}$, where C_m is a cyclic graph and $\overline{K_n}$ is an empty graph.

II. Pendant Total Domination Polynomial Of A Graph

Definition 2.1 Let G be a simple graph of order n with no isolated vertices. Let $d_{pet}(G,i)$ be the family of pendant total dominating sets of G with cardinality i and let $d_{pet}(G,i) = |D_{pet}(G)|$. Then the pendant total domination polynomial $d_{pet}(G,x)$ of G is defined as $d_{pet}(G,x) = \sum_{i=\gamma_{pet}(G)}^{n} d_{pet}(G,i)x^i$, Where $\gamma_{pet}(G)$ is the pendant total domination number of G.

Example 2.1 Consider the graph G given in the figure 2.1

(Figure 2.1)

The pendant total dominating sets of G of cardinality 5 is 1 i.e. $\{v_2, v_3, v_4, v_5, v_6\}$ .Therefore $d_{pet}(G,5)=1$.
The pendant total dominating sets of G of cardinality 6 is 2 i.e. $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\{v_2, v_3, v_4, v_5, v_6, v_7\}$. Therefore $d_{pet}(G,6)=2$.
The pendant total dominating sets of G of cardinality 7 is 1 i.e. $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Therefore $d_{pet}(G,7)=1$.
Since, the minimum cardinality is 5, $\gamma_{pet}(G)=5$

$$D_{pet}(G,x)=\sum_{i=\gamma_{pet}(G)}^{|V(G)|} d_{pet}(G,i)x^i$$

$$=\sum_{i=5}^{7} d_{pet}(G,i)x^i$$

$$= d_{pet}(G,5)x^5 + d_{pet}(G,6)x^6 + d_{pet}(G,7)x^7$$

$$= x^5 + 2x^6 + x^7$$

$$= x^5(1+2x+x^2)$$

$$D_{pet}(G,x) = x^5(1+x)^2$$

Observation 2.1 Let G be any connected graph of order $n \geq 2$. Then

a) $d_{pet}(G,n)=1$, if G has a pendant vertex.

b) $d_{pet}(G,n)=\begin{cases} 1, if\ \delta(G)=1 \\ 0, otherwise \end{cases}$

c) $d_{pet}(G,i)=0\ iff\ i<\gamma_{pet}(G)\ or\ i>n.$

d) $d_{pet}(G,x)$ has no constant term.

e) Zero is a root of $d_{pet}(G,x)$ of multiplicity $\gamma_{pet}(G)$.

Proposition 2.1 Let G be a connected graph of order greater than equal to 2 then $D_{pet}(G,x)=\binom{n}{2}x^2\ iff\ G \cong K_n$

Proof : Let G be a complete graph of order greater than equal to 2 . Clearly $\gamma_{pet}(G)=2$ and a pendant total dominating set of size two is obtained by choosing any two vertices in V(G). hence there are $\binom{n}{2}$ ways to select the pendant total dominating set of size two and so $D_{pet}(G,2)=\binom{n}{2}$. For any subset S of vertices, of size at least three, the induced sub graph <S> contains no pendant vertex and so $D_{pet}(G,i)=0$ for $i \geq 3$. Therefore $D_{pet}(G,x)=\binom{n}{2}x^2$.

Conversely, Let S be a γ_{pet} - set of G and assume $D_{pet}(G,x)=\binom{n}{2}x^2$. Since coefficient of x^i is zero for $i \geq 3$, it follows that G contains no pendant dominating set of size greater than two. Further , every pair of vertices will be a pendant dominating set and hence any two vertices in G are adjacent, proving that $G \cong K_n$.

Theorem 2 : Let G_1, G_2 be connected graphs of order $n,m \geq 2$ respectively. Then $D_{pet}(G_1 \vee G_2, x) = nmx^2 + D_{pet}(G_1,x) + D_{pet}(G_2,x)$.

Proof : Let $G \cong G_1 \vee G_2$ and suppose that D_1 is a pendant dominating set of G_1 and D_2 is a pendant dominating set of G_2. Further, $\gamma_{pet}(G)=2$ and there are $\binom{n}{1}\binom{m}{1}$ ways to select the minimum pendant dominating set in G. Next, selecting at least one vertex from G_1 and two vertices from G_2 (or conversely), then the resulting set leads to a sub graph having no pendant vertex. It implies that, $D_{pet}(G,i)=0, for\, i\geq 3$ and vertices taken from G_1 and G_2. Any pendant dominating set of size greater than or equal to three arises from or G_2. The pendant dominating sets of G_1 or G_2 will also be a pendant dominating set of G. Therefore

$$D_{pet}(G_1+G_2,x)=\left[\binom{n}{1}\binom{m}{1}\right]x^2+D_{pet}(G_1,x)+D_{pet}(G_2,x).$$

$$D_{pet}(G_1+G_2,x)=nmx^2+D_{pet}(G_1,x)+D_{pet}(G_2,x).$$

Corollary 2.1 For a connected graph G, the followings are true :

a) If $n\geq 4$, then $D_{pet}(W_n,x)=nx^2+D_{pet}(C_{n-1},x).$

b) If G is a m- gonal n- cone graph $C_{m,n}$, then $D_{pet}(C_{m,n},x)=nmx^2+D_{pet}(C_n,x).$

c) If G is a fan graph $F_{m,n}$, then $D_{pet}(F_{m,n},x)=nmx^2+D_{pet}(P_n,x).$

Proof : a) Let W_n be a wheel with $n\geq 4$ vertices, which can be constructed by joining the graph

$G_1=C_{n-1}\, with\, G_2=K_1, i.e., W_n=C_{n-1}+K_1$. Now by using Theorem 2, we get $D_{pet}(W_n,x)=nx^2+D_{pet}(C_{n-1},x).$

b) Let $G=C_{m,n}$ be a cone graph and it can be construct by joining the graph $G_1=C_m\, with\, G_2=\overline{K_n}, i.e., C_{m,n}=C_m+\overline{K_n}$. Now by using Theorem 2, we get $D_{pet}(C_{m,n},x)=nmx^2+D_{pet}(C_n,x).$

c) By applying the theorem 2, with $G_1=\overline{K_m}\, and\, G_2=P_n$, we have the result.

Proposition 2.2 The following properties holds for coefficient of $D_{pet}(P_n,x).$

a) $D_{pet}(P_n,n)=1, if\ n\geq 2.$

b) $D_{pet}(P_n,n-1)=n, if\ n\geq 3.$

c) $D_{pet}(P_n,n)=1,\ if\ n\geq 2.$

d) $D_{pet}(P_{3n},n)=0\ \forall n.$

Theorem 3 : Let $G\cong S_n$. Then $D_{pet}=x\left[(1+x)^n-1\right].$

Proof: Let $G\cong S_n$ be a star graph. Then $\gamma_{pet}(G)=2$ and $G\cong S_n\cong \overline{K_n}+K_1$. Let $\{u_1,u_2,u_3,.........,u_n\}$ be a vertex set of $\overline{K_n}$ and u be a vertex of K_1. Any pendant total dominating set in G must contain u and so the number of pendant total dominating sets of size two will be the number of totally disconnected graph and so $d_{pet}(G,2)=n.$

Also, pendant dominating sets of size $j, 3\leq j\leq n$ are obtained by taking any $j-1$ number of vertices from $V(\overline{K_n})$. Therefore, we have $\binom{n}{j-1}$ ways to choose a pendant set of j size and so $d_{pet}(G)=\binom{n}{j-1}.$

Therefore

$$D_{pet}(G,x)=\binom{n}{1}x^2+\binom{n}{2}x^3+\binom{n}{3}x^4+...+\binom{n}{n}x^{n+1}.$$

$$D_{pet}(G,x)=x\left[\binom{n}{1}x^1+\binom{n}{2}x^2+\binom{n}{3}x^3+...+\binom{n}{n}x^n\right].$$

$$D_{pet}(G,x)=x\left[\sum_{i=0}^{n}\binom{n}{i}x^i-1\right].$$

$$D_{pet}(G,x)=x\left[(1+x)^n-1\right].$$

Theorem 4 : Let $G \cong K_{n,n}$ be a complete bipartite graph. Then $D_{pet}(G,x) = (nx)^2\left[1+\frac{nx(1-x)-x}{(1-x)^2}\right]+2nx^{n+1}.$

Proof : Let G be a complete bipartite graph with 2n vertices. Clearly $\gamma_{pet}(G)=2.$ The minimum pendant total dominating set is of size two and maximum pendant total dominating set is of size $n+1$. There are n^2 edges of pendant total dominating set of size two. Therefore $d_{pet}(G,x)=n^2.$ Then there are $n^2\binom{n-1}{1}$ ways to select the pendant total dominating set of size three, similarly $n^2\binom{n-2}{1}$ ways to select the pendant total dominating set of size n and $\binom{2n}{1}$ ways to select the pendant total dominating set of size $n+1.$ Therefore

$$D_{pet}(G,x) = \left[n^2x^2+n^2(n-1)x^3+n^2(n-2)x^4+...\right]+\binom{2n}{1}x^{n+1}.$$

$$=(nx)^2\left[1+(n-1)x+(n-2)x^2+...\right]+2nx^{n+1}.$$

$$=(nx)^2\left[1+nx+nx^2+nx^3+...-x-2x^2-3x^3-...\right]+2nx^{n+1}.$$

$$=(nx)^2\left[1+nx(1+x+x^2+...)-x(1+2x+3x^2+...)\right]+2nx^{n+1}.$$

$$=(nx)^2\left[1+\frac{nx(1-x)-x}{(1-x)^2}\right]+2nx^{n+1}.$$

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