EXTENDED PSEUDO RICCI SYMMETRIC ADMITTING W_2-CURVATURE TENSOR

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Abstract: This paper deals with extended pseudo Ricci symmetric manifold with W_2 −curvature tensor. We determine several properties of these manifolds.

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1. Introduction: As a paper generalized of pseudo symmetric manifold by Chaki (1987), in 1989 Tamassy and Binh ( ) introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric if its curvature tensor satisfies the condition

$$(D_X R)(Y,Z,W,V) = A(X)R(Y,Z,W,V) + B(Y)R(X,Z,W,V) + C(Z)R(Y,X,W,V) + D(W)R(Y,Z,X,V) + E(V)R(Y,Z,W,X) \quad (1.1)$$

Equation (1.1) can be written as

$$(D_X R)(Y,Z,W) = A(X)R(Y,Z,W) + B(Y)R(X,Z,W) + C(Z)R(Y,X,W) + D(W)R(Y,Z,X) + g(R(Y,Z,W),X)\rho \quad (1.2)$$

for all vector fields $X, Y, Z, W, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (non simultaneously zero) and D denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind was denoted by $(WS)_n$.

In 1993, Tamassy and Binh introduced the notion of weakly Ricci symmetric manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor Ric of type (0,2) is not identically zero and satisfies the condition

$$(D_X Ric)(Y,Z) = A(X)Ric(Y,Z) + B(Y)Ric(X,Z) + C(Z)Ric(Y,X), \quad (1.3)$$

where A, B, C are three non-zero 1-forms, called the associated 1-forms of the manifold and D denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold was denoted by $(WRS)_n$. As an equivalent notion of $(WRS)_n$, Chaki and Koley (1994) introduced the notion of generalized pseudo Ricci symmetric manifold. If in (1.3) the 1-form A is replaced by 2A then the definition of $(WRS)_n$ reduces to that of generalized pseudo Ricci symmetric manifolds by $(G(PRS)_n)$ and is given by

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + B(Y)Ric(X,Z) + C(Z)Ric(Y,X), \quad (1.4)$$

Either in (1.3) the 1-form A is replaced by $2A$ and B and C replaced by A or in (\1.4) B and C replaced by A then then the manifold is called pseudo Ricci symmetric manifold in the sense of Chaki (1988) and is given by

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(Y,X), \qquad (1.5)$$

denoted by $(PRS)_n$.

Also De and Sengupta (1999) proved that if a $(WS)_n$ admits a type of semi-symmetric metric connection then the $(WS)_n$ reduces to

$$(D_X Ric)(Y,Z) = A(X)Ric(Y,Z) + B(Y)Ric(X,Z) + B(Z)Ric(Y,X), \qquad (1.6)$$

i.e. the $(WS)_n$ reduces to a special type of $(WRS)_n$. The geometry of $(WS)_n$, $(WRS)_n$, $(PRS)_n$ (and $G(PRS)_n$ have been studied by many geometers in a different structures.

Especially if $A = B = C = 0$ then $(WRS)_n$, $G(PRS)_n$ and $(PRS)_n$ reduces to Ricci-symmetric or covariantly constant i.e.

$$(D_X Ric)(Y,Z) = 0 \qquad (1.7)$$

The notion of pseudo Ricci symmetric is different from that of Deszez (1992). Therefore, the pseudo Ricci symmetric manifolds have some importance in general theory of relativity. By this motivation, Prasad and Doulo (2016) generalized the pseudo Ricci symmetric manifolds and introduced the notion of extended pseudo Ricci-symmetric manifolds as

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + B(Y)Ric(X,Z) + B(Z)Ric(Y,X), \qquad (1.8)$$

where A and B are two non-zero and D has the meaning already mentioned. In such a case A and B are called the associated 1-forms such that $g(X,\rho_1) = A(X)$, $g(X,\rho_2) = B(X)$\$ for all X and n-dimensional manifold of this kind was denoted by $E(PRS)_n$.

If in particular $A = B$, then (1.8) takes the form of Chaki's $(PRS)_n$ as in (1.5).

If in particular $A = 0$, then (1.8) takes the form

$$(D_X Ric)(Y,Z) = B(Y)Ric(X,Z) + B(Z)Ric(Y,X), \qquad (1.9)$$

This Ricci tensor has been studied and defined by Tarafdar and Jawarneh in 1993 and called by them semi-pseudo Ricci symmetric n-dimensional manifold and denoted by $(SPRS)_n$. Moreover, if we replace B by $-A$\$ in (1.8), then it becomes

$$(D_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) - A(Y)Ric(X,Z) - A(Z)Ric(Y,X), \qquad (1.10)$$

This Ricci tensor investigated by Mukhopadhyay and Barua in 1990-91 and this structure were called as semi-Ricci symmetric manifold and denoted by $(SRS)_n$. Also, if we replace $2A$ by A in (1.8), then this takes special type $(WRS)_n$ in the sence of De and Sengupta (1999) as given by (1.6).

Finally if $A(X) = 0$ and $B(Y)$ is replaced by $C(Y)$ and $B(Z)$ is replaced by $D(Y)$ in (1.8) Then it gives

$$(D_X Ric)(Y,Z) = B(Y)Ric(X,Z) + D(Z)Ric(Y,X), \qquad (1.11)$$

This type of Ricci tensor established by Jawarneh and Tashtoush in 2012 and called by them generalized semi-pseudo Ricci symmetric manifold and denoted by $G(SPRS)_n$. This justifies the name ``extended pseudo Ricci symmetric manifold" defined by (1.8) and use the symbol $E(PRS)_n$ for it.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called generalized Ricci recurrent if its Ricci tensor Ric satisfies the condition (1995)

$$(D_Z\, Ric)(X,Y) = A(Z)Ric(X,Y) + B(Z)g(X,Y), \quad (1.12)$$

where A and B are non-zero 1-forms. If the associated 1-forms \$B\$ becomes zero, then the manifold becomes Ricci recurrent (1952), that is

$$(D_Z\, Ric)(X,Y) = A(Z)Ric(X,Y)\,, \quad (1.13)$$

In a Riemannian manifold the Ricci tensor is called Codazzi type if the following condition holds

$$(D_Z\, Ric)(X,Y) = (D_X\,\mathrm{Ric})(\mathrm{Y},\mathrm{Z}) \quad (1.14)$$

Pokhariyal and Mishra (2010) introduced W_2 −curvature tensor on a Riemannian manifold (M^n, g) $(n > 2)$ by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \quad (1.15)$$

where $Ric(X,Y) = g(QX,Y), Q$ be the symmetric endomorphism corresponding to the Ricci tensor. The W_2 −curvature tensor on some special manifolds has been examined by many authors as Prasad (1997) , Praksha (14), Malick and De (1914), Zengin and Bektas (2019), Hui (2012) etc .

The object of the present paper is to study W_2 −Ricci tensor in $E(PRS)_n$ and obtained various geometrical properties.

2. $E(PRS)_n$ with $\overline{W}_2$ −flat Ricci tensor: Contracted W_2 −curvature tensor by $W_2(X,Y)$ and call it $\overline{W}_2$ −Ricci tensor. That is, W_2 −Ricci tensor

$$\overline{W}_2(X,Y) = \frac{n}{n-1}\left[Ric(X,Y) - \frac{r}{n}g(X,Y)\right] \quad (2.1)$$

Here we assume manifold is W_2 −flat. Hence from (2.1), we get

$$Ric(X,Y) = \frac{r}{n}g(X,Y) \quad (2.2)$$

From (2.2), we get

$$(D_Z Ric\}(X,Y) = \frac{D_Z r}{n}g(X,Y) \quad (2.3)$$

Here we assume that manifold is \$$E(PRS)_n$. Hence we have from (1.8)

$$(D_Z Ric)(X,Y) = 2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) + B(Y)Ric(X,Z), \quad (2.4)$$

Hence in view of (2.4), (2.2) and (2.3), we get

$$(D_Z r)g(X,Y) = r[2A(Z)g(X,Y) + B(X)g(Z,Y) + B(Y)g(X,Z)] \quad (2.5)$$

Contracting (2.5) with respect to X and Y, we get

$$n(D_Z r) = 2r[nA(Z) + B(Z)] \quad (2.6)$$

Contraction of (2.5) gives

$$(D_Z r) = r[2A(Z) + (n+1)B(Z)] \quad (2.7)$$

From (2.6) and (2.7), we get

$$r(n^2 + n - 2)B(Z) = 0 \quad (2.8)$$

From (2.8), we see that $r \neq 0$, we have $B(Z) = 0$.

Thus in view of (1.12), we have the following theorem:

Theorem (2.1): $W_2 - flat\ E(PRS)_n$ reduces to Ricci recurrent manifold with vector field generated by the 1-form A.

3. $E(PRS)_n$ with $\overline{W}_2$ −Ricci tensor as conservative:

In this section we assume that $\overline{W}_2$ is conservative. That is

$$div\overline{W}_2 = 0. \quad (3.1)$$

Now differentiating (2.1) covariantly, we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}[(D_ZRic)(X,Y) - \frac{D_Zr}{n}g(X,Y)] \quad (3.2)$$

Contracting (2.4}) over X and Y, we get

$$(D_Zr) = 2A(Z)r + 2B(QZ) \quad (3.3)$$

From (2.4), (3.2) and (3.3), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}[2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) + B(Y)Ric(X,Z)$$
$$-\frac{1}{n}\{2A(Z)r + 2B(QZ)\}g(X,Y)] \quad (3.4)$$

which gives

$$(div\overline{W}_2)(Y) = \frac{n}{n-1}\left[2A(QY) - \frac{2r}{n}A(Y) + \frac{n-2}{n}B(QY) + B(Y)r\right] \quad (3.5)$$

In view of (3.1) and (3.5), we get

$$2A(QY) - \frac{2r}{n}A(Y) + \frac{n-2}{n}B(QY) + B(Y)r = 0. \quad (3.6)$$

Hence we can state the following theorem:

Theorem(3.1): For an $E(PRS)_n$, tensor $\overline{W}_2$ be conservative if and only if $\frac{r}{n}$ and $\frac{2-n}{n}$ $(n > 2)$ be the eigen values of the Ricci tensor Ric corresponding to the eigen vectors ρ_1 and ρ_2 where $g(X,\rho_1) = A(X)$ and $g(X,\rho_2) = B(X)$ respectively.

4. $E(PRS)_n$ with $\overline{W}_2$ − tensor as a Ricci-recurrent and generalized Ricci-recurrent:}

In this section we assume that $\overline{W}_2$ be Ricci recurrent. Then from (1.13), we get

$$(D_Z\overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y). \quad (4.1)$$

\end{equation}

where α is 1-form.

Thus in view of (2.1), (3.4) and (4.1), we get

$$\alpha(Z)\left[Ric(X,Y) - \frac{r}{n}g(X,Y)\right] = [2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) + B(Y)Ric(X,Z)$$
$$-\frac{2r}{n}.A(Z)g(X,Y) - \frac{2}{n}B(QZ)g(X,Y)] \quad (4.2)$$

Contracting (4.2), we get

$$\alpha(QZ) - \frac{r}{n}\alpha(Z) = 2A(QZ) - \frac{2r}{n}.A(Z) - \frac{n-2}{n}B(QZ) + B(Z)r. \tag{4.3}$$

If we take $\alpha(Z) = A(Z)$ in (4.3), we get

$$2A(QZ) - \frac{2r}{n}.A(Z) = \frac{2-n}{n}B(QZ) - B(Z)r. \tag{4.4}$$

Hence from (4.4), we have the following theorem:

Theorem (4.1): In an $E(PRS)_n$,, if $\overline{W}_2$ −Ricci tensor is Ricci recurrent with the recurrent vector generated by the 1-form A, then a necessary and sufficient condition for $\frac{r}{n}$ be an eigen value of the Ricci tensor Ric corresponding to the eigen vector ρ_1 where $g(X,\rho_1) = A(X)$ is that $\frac{n-2}{n}$ $(n > 2)$ be the eigen values of the Ricci tensor Ric corresponding to the eigen vector ρ_2 where $g(X,\rho_2) = B(X)$.

Further we assume that $\overline{W}_2$ −tensor be generalized Ricci recurrent. Then from (1.12), we get

$$(D_Z\overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y) + \gamma(Z)g(X,Y). \tag{4.5}$$

In view of (3.1), (4.3) and (4.5), we get

$$\frac{n}{n-1}.\alpha(Z)\left[Ric(X,Y) - \frac{r}{n}g(X,Y)\right] + \gamma(Z)g(X,Y) = \frac{n}{n-1}[2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) + B(Y)Ric(X,Z) - \frac{2r}{n}.A(Z)g(X,Y) - \frac{2}{n}B(QZ)g(X,Y)] \tag{4.6}$$

Contracting over X and Y, we get

$$\gamma(Z) = 0 \tag{4.7}$$

Thus we can state the following theorem:

Theorem (4.2): An $E(PRS)_n$with $\overline{W}_2$ −Ricci tensor cannot be generalized Ricci recurrent .

5. $E(PRS)_n$with its $\overline{W}_2$ −Ricci tensor as a Codazzi type:

Now we assume that r is a non--zero constant then from (3.2), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}(D_ZRic)(X,Y) \tag{5.1}$$

From (2.4) and (5.1), we get

$$(D_Z\overline{W}_2)(X,Y) = \frac{n}{n-1}[2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) + B(Y)Ric(X,Z)] \tag{5.2}$$

Let $\overline{W}_2$ is Codazzi type Ricci tensor, then we have

$$(D_Z\overline{W}_2)(X,Y) = (D_X\overline{W}_2)(Y,Z) \tag{5.3}$$

In view of (5.2) and (5.3), we get

$$2A(Z)Ric(X,Y) + B(X)Ric(Z,Y) = 2A(X)Ric(Y,Z) + B(Z)Ric(Y,X) \tag{5.4}$$

Contracting equation (5.4) with respect X and Y, we get

$$2A(QZ) - 2A(Z)r = B(QZ) - B(Z)r . \tag{5.5}$$

Hence from (5.5), we can state the following theorem:

Theorem (5.1): For an $E(PRS)_n$ constant scalar curvature, if $\overline{W}_2$ −Ricci tensor is Codazzi type then a necessary and sufficient condition for r be an eigen value of the Ricci tensor Ric corresponding to the eigen vector ρ_2 where $g(X, \rho_2) = B(X)$ is that r be an eigen values of the Ricci tensor Ric corresponding to the eigen vector ρ_1 where $g(X, \rho_1) = A(X)$.

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