Energy Comparison of Unicyclic Graphs with Cycle C_4

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Abstract

In this paper, we compare the energies of unicycilc graphs with cycle C_4, (with k number of vertices, having the unique cycle C_4, denoted by $G''_{i,k}$), using the coefficients of the characteristic polynomials and Coulson integral formula by establishing the quasi- ordering '≤' on the unicyclic graphs of same order k.

Keywords. Energy; Characteristic polynomial; Adjacency matrix $A(G)$; Unicyclic graphs; Bipartite graphs.

1. Introduction

Let G be a simple graph with k vertices and $A(G)$ be its adjacency matrix. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of $A(G)$. Then the *energy* of G, denoted by $E(G)$, is defined as $E(G) = \sum_{i=1}^{k} |\lambda_i|$.

The characteristic polynomial $\det(xI - A(G))$ of the adjacency matrix $A(G)$ of the graph G is also called the *characteristic polynomial* of G is written as

$$\phi(G, x) = \sum_{i=0}^{k} a_i(G)\; x^{k-i}$$

Using the coefficients $a_i(G)$ of $\phi(G, x)$, the energy $E(G)$ of the graph G with k vertices can be expressed by the following Coulson integral formula (Eq. (3.11) in [2]):

$$E(G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log\left[\left(\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i a_{2i}(G)\, x^{2i}\right)^2 + \left(\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i a_{2i+1}(G)\, x^{2i+1}\right)^2\right] \mathrm{d}x.$$

We write $b_i(G) = |a_i(G)|$. Then clearly $b_0(G) = 1$, $b_1(G) = 0$ and $b_2(G)$ equals the number of edges of G.

About the signs of the coefficients of the characteristic polynomials of unicyclic graphs, we have the following result:

Lemma 1.1: (Lemma 1 in [3]) Let G be a unicyclic graph and the length of the unique cycle of G be `. Then we have the following:

(1) $b_{2i}(G) = (-1)^i a_{2i}(G)$,

(2) $b_{2i+1}(G) = (-1)^i a_{2i+1}(G)$, if G contain a cycle of length l with $l \equiv 1 (\mathrm{mod}\ 4)$,

(3) $b_{2i+1}(G) = (-1)^{i+1} a_{2i+1}(G)$, if G contain a cycle of length l with $l \not\equiv 1 (\mathrm{mod}\ 4)$.

Thus, the Coulson integral formula for unicyclic graphs can be rewritten in terms of $b_i(G)$ as follows:

$$E(G) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{x^2}\log\left[\left(\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor} b_{2i}(G)\, x^{2i}\right)^2 + \left(\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor} b_{2i+1}(G)\, x^{2i+1}\right)^2\right]\mathrm{d}x.$$

Hence it follows that for unicyclic graphs G, $E(G)$ is a strictly monotonically increasing function of $b_i(G)$, $i = 0, \ldots, k$. To make it more precise, we define a quasi-order $\preceq$ on graphs as follows:

Definition 1.2: Let G_1 and G_2 be two graphs of order k. If $b_i(G_1) \le b_i(G_2)$ for all i with $1 \le i \le k$, then we write $G_1 \le G_2$.

Thus using Coulson integral formula, we have,

Theorem 1.3: For any two unicyclic graphs G_1 and G_2 of order k, we have,

$$G_1 \le G_2 \Longrightarrow E(G_1) \le E(G_2).$$

Thus, for comparing the energies of any two unicyclic graphs of the same order, it is enough to establish the quasi-order.

Using this idea, in Section 2, we compare the energies of the *unicyclic graphs* $G''_{1,k}$, $G''_{2,k}$, $G''_{3,k}$ and $G''_{4,k}$ (see Fig. 2.1, Fig. 2.2, Fig. 2.3 and Fig. 2.4) with k vertices having the unique cycle C_4. These graphs are comparable with respect to the quasi-ordering:

$$G''_{2,k} \le G''_{4,k} \le G''_{3,k} \le G''_{1,k}.$$

We note that these graphs are *bipartite*. For a bipartite graph G, the characteristic polynomial is of the form

$$\phi(G,x) = \sum_{i=0}^{k} a_i \; x^{k-i} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} a_{2j} \; x^{k-2j},$$

as $a_{2j+1} = 0$ for $j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$. Also, $(-1)^j a_{2j} = b_{2j}$ and so

$$\phi(G,x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j b_{2j} x^{k-2j}.$$

Thus, for a bipartite graph G, the Coulson integral formula

reduces to

$$E(G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x} \log\left[\left(\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j a_{2j}(G) \; x^{k-2j}\right)^2\right] \mathrm{d}x$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x} \log\left[\left(\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} b_{2j}(G) \; x^{k-2j}\right)^2\right] \mathrm{d}x,$$

from which the monotonicity of the $E(G)$ with respect to the $b_i(G)$, $1 \leq i \leq k$, follows. i.e., if G_1 and G_2 are two bipartite graphs of order k such that $b_i(G_1) \leq b_i(G_2)$ for all i, $1 \leq i \leq k$, then $E(G_1) \leq E(G_2)$. i.e., $G_1 \leq G_2$ implies $E(G_1) \leq E(G_2)$.

2. Comparing the energies of the graphs $G''_{i,k}$

We consider the unicyclic graphs $G''_{1,k}$, $G''_{2,k}$, $G''_{3,k}$, $G''_{4,k}$ (see Fig. 2.1, Fig. 2.2, Fig. 2.3 and Fig. 2.4) with unique cycle C_4. These graphs are clearly bipartite. We use Theorem 1.3 to compare their energies using Coulson integral formula by establishing the partial order $G''_{1,k} \leq G''_{2,k} \leq G''_{3,k} \leq G''_{4,k}$.

Theorem 2.1: Let $G''_{1,k}$ be the graph with k vertices given below:

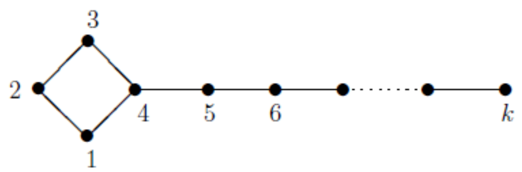

Fig. 2.1 Graph $G''_{1,k}$

Let $A(G''_{1,k})$ be the adjacency matrix of the graph $G''_{1,k}$. Then, for $k \geq 7$, its characteristic polynomial $\chi(G''_{1,k})$ is given by:

$$\chi(G''_{1,k}) = \lambda\, \chi(P_{k-1}) - \chi(P_{k-2}) - (\lambda^2+1)\, \chi(P_{k-4}) \quad (1$$

Also, for $1 \le i \le k$, the coefficient of λ^i in $\chi(G''_{1,k})$ is

$$(-1)^{\frac{k-i}{2}} \left[\binom{\frac{k+i-2}{2}}{\frac{k-i}{2}} + \binom{\frac{k+i-2}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} - \binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}} \right] \quad (2$$

Proof: The adjacency matrix $A(G''_{1,k})$ is given by

$$A(G''_{1,k}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_k$$

The characteristic polynomial of the adjacency matrix $A(G''_{1,k})$ is given by $\chi(G''_{1,k}) = |\lambda I - A|$, where I is the identity matrix of order k. Thus, by expanding the following determinant and the subsequent determinants by their first column, we get,

$$\chi(G''_{1,k}) = \begin{vmatrix} \lambda & -1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & \lambda & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \lambda & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & \lambda & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda \end{vmatrix}_k$$

$$= \lambda \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & \cdots & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-1} + \begin{vmatrix} -1 & 0 & -1 & \cdots & 0 \\ -1 & \lambda & -1 & \cdots & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-1} + \begin{vmatrix} -1 & 0 & -1 & \cdots & 0 \\ \lambda & -1 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-1}$$

$$= \lambda\, \chi(P_{k-1}) + \left\{ (-1) \begin{vmatrix} \lambda & -1 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-2} + \begin{vmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-2} \right\}$$

$$-\left\{\begin{vmatrix}-1 & 0 & 0 & \dots & 0\\ \lambda & -1 & 0 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-2} + \lambda\begin{vmatrix}0 & -1 & 0 & \dots & 0\\ \lambda & -1 & 0 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-2} + \begin{vmatrix}0 & -1 & 0 & \dots & 0\\ -1 & 0 & -1 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-2}\right\}$$

$$= \lambda\,\chi(P_{k-1}) - \chi(P_{k-2}) + \begin{vmatrix}\lambda & -1 & 0 & \dots & 0\\ -1 & \lambda & -1 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-3} + \begin{vmatrix}-1 & 0 & 0 & \dots & 0\\ -1 & \lambda & -1 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-3}$$

$$+\lambda^2\begin{vmatrix}-1 & 0 & 0 & \dots & 0\\ -1 & \lambda & -1 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-3} - \begin{vmatrix}-1 & 0 & 0 & \dots & 0\\ -1 & \lambda & -1 & \dots & 0\\ 0 & -1 & \lambda & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & \lambda\end{vmatrix}_{k-3}$$

$$\begin{aligned} &= \lambda\,\chi(P_{k-1}) - \chi(P_{k-2}) - \chi(P_{k-4}) - \chi(P_{k-4}) - \lambda^2\chi(P_{k-4}) + \chi(P_{k-4})\\ &= \lambda\,\chi(P_{k-1}) - \chi(P_{k-2}) - (\lambda^2+1)\,\chi(P_{k-4}),\end{aligned}$$

where $\chi(P_{k-1})$, $\chi(P_{k-2})$ and $\chi(P_{k-4})$ denote the characteristic polynomials of the paths P_{k-1}, P_{k-2} and P_{k-4} containing $k-1$, $k-2$ and $k-4$ vertices respectively.

This proves (1).

We now compute the coefficient of λ^i in $\chi(G''_{1,k})$ using (1). We make use of the following characteristic polynomial of the path P_n:

$$\chi(P_n) = \sum_{t=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^t\binom{n-t}{t}\lambda^{n-2t} \qquad (3$$

Put $n = k - 1$ in (3). If $t = \frac{k-i}{2}$, then $n - 2t = i - 1$, and so the coefficient of λ^{i-1} in $\chi(P_{k-1})$ is $(-1)^{\frac{k-i}{2}}\binom{\frac{k+i-2}{2}}{\frac{k-i}{2}}$, since then $n - t = k - 1 - \left(\frac{k-i}{2}\right) = \frac{k+i-2}{2}$. Also, by putting $n = k - 2$ in (3) and taking $t = \frac{k-i-2}{2}$, we obtain the coefficient of

λ^i in $\chi(P_{k-2})$ to be $(-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-2}{2}}{\frac{k-i-2}{2}}$, since $n - 2t = k - 2 - 2\left(\frac{k-i-2}{2}\right) = i$ and

$n - t = k - 2 - \left(\frac{k-i-2}{2}\right) = \frac{k+i-2}{2}$.

Similarly, by putting $n = k - 4$ in (3) and taking $t = \frac{k-i-2}{2}$, we get the coefficient of λ^{i-2} in $\chi(P_{k-4})$ to be $(-1)^{\frac{k-i-2}{2}} \binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}$. Further putting $n = k-4$ and taking $t = \frac{k-i-4}{2}$ in (3), we see that the coefficient of λ^i in $\chi(P_{k-4})$ is $(-1)^{\frac{k-i-4}{2}} \binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}$

Now by (1), we have,

{Coefficient of λ^i in $\chi(G''_{1,k};\lambda)$}=

{Coefficient of λ^{i-1} in $\chi(P_{k-1})$} − {Coefficient of λ^i in $\chi(P_{k-2})$} − {Coefficient of λ^{i-2} in $\chi(P_{k-4})$} − {Coefficient of λ^i in $\chi(P_{k-4})$}.

Thus the coefficient of λ^i in $\chi(G''_{1,k};\lambda)$ is given by

$$(-1)^{\frac{k-i}{2}}\binom{\frac{k+i-2}{2}}{\frac{k-i}{2}} - (-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-2}{2}}{\frac{k-i-2}{2}} - (-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} - (-1)^{\frac{k-i-4}{2}}\binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}$$
$$= (-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-2}{2}}{\frac{k-i}{2}} + \binom{\frac{k+i-2}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} - \binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}\right].$$

Hence the theorem.

Theorem 2.2: Let $G''_{2,k}$ be the graph with k vertices given below:

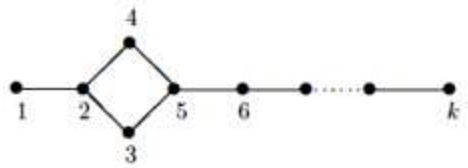

Fig. 2.2 Graph $G''_{2,k}$

Let $A(G''_{2,k})$ be the adjacency matrix of the graph $G''_{2,k}$. Then, for $k \geq 7$, its characteristic polynomial $\chi(G''_{2,k})$ is given by:

$$\chi(G''_{2,k}) = (\lambda^3 - 2\lambda)\ \chi(P_{k-3}) - \lambda^2\ \chi(P_{k-4}) - \lambda^3\ \chi(P_{k-5}) \qquad (4$$

Also, for $1 \leq i \leq k$, the coefficient of λ^i in $\chi(G''_{2,k})$ is

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-6}{2}}{\frac{k-i}{2}} + 2\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}\right] \qquad (5$$

Proof: The adjacency matrix $A(G''_{2,k})$ is given by

$$A(G''_{2,k}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_k .$$

The characteristic polynomial of the adjacency matrix $A(G''_{2,k})$ is given by $\chi(G''_{2,k}) = |\lambda I - A|$, where I is the identity matrix of order k. Thus

$$\chi(G''_{2,k}) = \begin{vmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & \lambda & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \lambda & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \lambda & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -1 & \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda \end{vmatrix}_k$$

$$= \lambda \begin{vmatrix} \lambda & -1 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & \lambda & 0 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 0 & \lambda & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & -1 & \lambda & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & \lambda & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \lambda \end{vmatrix}_{k-1} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & \lambda & 0 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 0 & \lambda & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & -1 & \lambda & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & \lambda & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \lambda \end{vmatrix}_{k-1}$$

$$= \lambda^2 \begin{vmatrix} \lambda & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -1 & \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-2} + \lambda \begin{vmatrix} -1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-2}$$

$$- \lambda \begin{vmatrix} -1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ \lambda & 0 & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-2} + (-1) \begin{vmatrix} \lambda & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-2}$$

$$= \lambda^2 \left\{ \lambda \begin{vmatrix} \lambda & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-3} - \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ \lambda & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-3} \right\}$$

$$-\lambda\left\{\begin{vmatrix} \lambda & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-3} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \lambda & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{k-3}\right\}$$

$$-\lambda\left\{(-1)\begin{vmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-3} - \lambda\begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-3} - \begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-3}\right\}$$

$$-\left\{\lambda\begin{vmatrix} \lambda & -1 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-3} - \begin{vmatrix} 0 & -1 & 0 & \dots & 0 \\ \lambda & -1 & 0 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}_{k-3}\right\}$$

$= \lambda^3 \chi(P_{k-3}) - \lambda^3 \chi(P_{k-5}) - \lambda\, \chi(P_{k-3}) - \lambda\, \chi(P_{k-5}) - \lambda\, \chi(P_{k-5}) - \lambda^2 \chi(P_{k-4})$

$+\lambda\, \chi(P_{k-5}) - \lambda\, \chi(P_{k-3}) + \lambda\, \chi(P_{k-5})$

$= (\lambda^3 - 2\lambda)\, \chi(P_{k-3}) - \lambda^2 \chi(P_{k-4}) - \lambda^3 \chi(P_{k-5})$,

proving (4).

To compute the coefficient of λ^i in $\chi(G''_{2,k})$, we first compute the coefficients of λ^{i-3} and λ^{i-1} in $\chi(P_{k-3})$. Put $n = k - 3$ in (3). If $t = \frac{k-i}{2}$ then $n - 2t = i - 3$, and so the coefficient of λ^{i-3} in $\chi(P_{k-3})$ is $(-1)^{\frac{k-i}{2}}\binom{\frac{k+i-6}{2}}{\frac{k-i}{2}}$, since $n - t = k - 3 - \left(\frac{k-i}{2}\right) = \frac{k+i-6}{2}$. Again by putting $n = k-3$ in (3), we obtain the coefficient of λ^{i-1} in $\chi(P_{k-3})$ to be $(-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}}$ by taking $t = \frac{k-i}{2}$ as $n - 2t = k - 3 - 2\left(\frac{k-i-2}{2}\right) = i - 1$ and $n - t = k - 3 - \left(\frac{k-i-2}{2}\right) = \frac{k+i-4}{2}$. Similarly, by putting $n = k - 4$ and taking $t = \frac{k-i-2}{2}$ in (3), we see that the coefficient of λ^{i-2} in $\chi(P_{k-4})$ to be $(-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}$. Further, putting $n = k-5$ in (3), we see that the coefficient of λ^{i-3} in $\chi(P_{k-5})$ is $(-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}$ by taking $t = \frac{k-i-2}{2}$.

Now by (4), we have,

{Coefficient of λ^i in $\chi(G''_{2,k})$}=

{Coefficient of λ^{i-3} in $\chi(P_{k-3})$} − 2{Coefficient of λ^{i-1} in $\chi(P_{k-3})$} − {Coefficient of λ^{i-2} in $\chi(P_{k-4})$} − {Coefficient of λ^{i-3} in $\chi(P_{k-5})$}.

Thus, the coefficient of λ^i in $\chi(G''_{2,k})$, is given by

$$(-1)^{\frac{k-i}{2}}\binom{\frac{k+i-6}{2}}{\frac{k-i}{2}} - 2(-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}} - (-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} - (-1)^{\frac{k-i-2}{2}}\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}$$

$$= (-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-6}{2}}{\frac{k-i}{2}} + 2\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}\right].$$

Hence the theorem.

Theorem 2.3: Let $G''_{3,k}$ be the graph with k vertices given below:

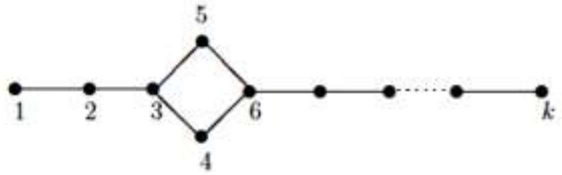

Fig. 2.3 Graph $G''_{3,k}$

Let $A(G''_{3,k})$ be its adjacency matrix. Then, for $k \geq 7$, the characteristic polynomial of the graph $G''_{3,k}$ is given by:

$$\chi(G''_{3,k}) = \lambda\,\chi(G''_{2,k-1}) - \chi(G''_{1,k-2}) \quad (6$$

Also, for $1 \leq i \leq k$, the coefficient of λ^i in $\chi(G''_{3,k})$} is

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i}{2}} + 2\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}} + \binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}} - \binom{\frac{k+i-6}{2}}{\frac{k-i-6}{2}}\right] \quad (7)$$

Proof: Proof of (6) follows from Theorem 2.2. Since we have, $\chi(G''_{3,k},\lambda) = \lambda\,\chi(G''_{2,k-1}) - \chi(G''_{1,k-2})$, the coefficient of λ^i in $\chi(G''_{3,k})$ is

{Coefficient of λ^{i-1} in $\chi(G''_{2,k-1})$}-{Coefficient of λ^i in $\chi(G''_{1,k-2})$.

By replacing i by $i-1$ and k by $k-1$ in (5), the coefficient of λ^{i-1} in $\chi(G''_{2,k-1})$is seen to be:

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i}{2}} + 2\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}\right].$$

Also, by replacing k by $k-2$ in (2), we obtain the coefficient of λ^i in $\chi(G''_{1,k-2})$to be:

$$(-1)^{\frac{k-i-2}{2}}\left[\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}} + \binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}} + \binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}} - \binom{\frac{k+i-6}{2}}{\frac{k-i-6}{2}}\right].$$

Thus, the coefficient of λ^i in $\chi(G''_{3,k})$ is given by,

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i}{2}}+2\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}\right]$$

$$-(-1)^{\frac{k-i-2}{2}}\left[\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-6}{2}}{\frac{k-i-6}{2}}\right]$$

$$=(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i}{2}}+2\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}\right]+(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}}\right.$$
$$\left.+\binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-6}{2}}{\frac{k-i-6}{2}}\right],$$

$$=(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i}{2}}+2\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-4}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-4}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}\right.$$
$$\left.+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-6}{2}}{\frac{k-i-6}{2}}\right].$$

Hence the Theorem.

Theorem 2.4: Let $G''_{4,k}$ be the graph with k vertices given below:

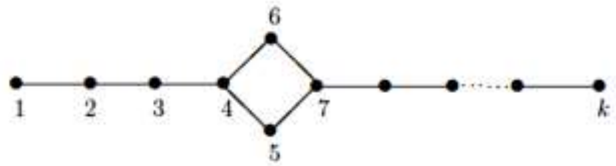

Fig. 2.4 Graph $G''_{4,k}$

Let $A(G''_{4,k})$ be its adjacency matrix. Then, for $k \geq 7$, the characteristic polynomial of the graph $G''_{4,k}$ is given by:

$$\chi(G''_{4,k}) = \lambda\, \chi(G''_{3,k-1}) - \chi(G''_{2,k-2})$$

Also, for $1 \leq i \leq k$, the coefficient of λ^i in $\chi(G''_{4,k})\}$ is

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-10}{2}}{\frac{k-i}{2}}+3\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+3\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}\right.$$
$$\left.+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-12}{2}}{\frac{k-i-2}{2}}+2\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-8}{2}}{\frac{k-i-6}{2}}\right].$$

Proof: Proof of (8) follows from Theorem 2.2. Since we have, $\chi(G''_{4,k}, \lambda) = \lambda\, \chi(G''_{3,k-1}) - \chi(G''_{2,k-2})$, the coefficient of λ^i in $\chi(G''_{4,k})$ is

{Coefficient of λ^{i-1} in $\chi(G''_{3,k-1})\}$ – {Coefficient of λ^i in $\chi(G''_{2,k-2})\}$.

By putting $k = k-1$, $i = i-1$ in (7), the coefficient of λ^{i-1} in $\chi(G''_{3,k-1})$ is seen to be:

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-10}{2}}{\frac{k-i}{2}}+2\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-12}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-8}{2}}{\frac{k-i-6}{2}}\right].$$

Also, replacing k by $k-2$ in (5), we obtain the coefficient of λ^i in $\chi(G''_{2,k-2})$) to be:

$$(-1)^{\frac{k-i-2}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+2\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}\right].$$

Thus the coefficient of λ^i in $\chi(G''_{4,k})$ is given by,

$$(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-10}{2}}{\frac{k-i}{2}}+2\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-12}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-8}{2}}{\frac{k-i-6}{2}}\right]-(-1)^{\frac{k-i-2}{2}}\left[\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+2\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}\right]$$

$$=(-1)^{\frac{k-i}{2}}\left[\binom{\frac{k+i-10}{2}}{\frac{k-i}{2}}+3\binom{\frac{k+i-8}{2}}{\frac{k-i-2}{2}}+3\binom{\frac{k+i-6}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-6}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-10}{2}}{\frac{k-i-2}{2}}+\binom{\frac{k+i-8}{2}}{\frac{k-i-4}{2}}+\binom{\frac{k+i-12}{2}}{\frac{k-i-2}{2}}+2\binom{\frac{k+i-10}{2}}{\frac{k-i-4}{2}}-\binom{\frac{k+i-8}{2}}{\frac{k-i-6}{2}}\right].$$

This proves (9).

Theorem 2.5: For the graphs $G''_{1,k}$, $G''_{2,k}$, $G''_{3,k}$ and $G''_{4,k}$, we have, for $k \geq 10$,

$$b_i(G''_{1,k}) \geq b_i(G''_{3,k}) \geq b_i(G''_4) \geq b_i(G''_{2,k})$$

for all i.

Proof: We prove that:

(i) $b_i(G''_{1,k}) \geq b_i(G''_{3,k})$,

(ii) $b_i(G''_{3,k}) \geq b_i(G''_{4,k})$,

(iii) $b_i(G''_{4,k}) \geq b_i(G''_{2,k})$.

Proof of (i): By putting $\frac{k+i}{2}=r$ and $\frac{k-i}{2}=s$ in (2) and (7), we obtain,

$$b_i(G''_{1,k})=\left[\binom{r-1}{s}+\binom{r-1}{s-1}+\binom{r-3}{s-1}-\binom{r-2}{s-2}\right]$$

and

$$b_i(G''_{3,k})=\left[\binom{r-4}{s}+2\binom{r-3}{s-1}+\binom{r-2}{s-2}-\binom{r-2}{s-1}+\binom{r-4}{s-1}+\binom{r-5}{s-1}+\binom{r-4}{s-2}-\binom{r-3}{s-3}\right].$$

By using the binomial identity, we have,

$b_i(G''_{1,k})$

$$= \binom{r-1}{s} + \binom{r-1}{s-1} + \binom{r-3}{s-1} - \binom{r-2}{s-2}$$

$$= \left[\binom{r-2}{s} + \binom{r-2}{s-1}\right] + \left[\binom{r-2}{s-1} + \binom{r-2}{s-2}\right] + \left[\binom{r-4}{s-1} + \binom{r-4}{s-2}\right] - \binom{r-2}{s-2}$$

$$= \left[\binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2}\right] + \binom{r-2}{s} + 2\binom{r-2}{s-1} - \binom{r-2}{s-2}$$

$$= \left[\binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2}\right] + \left[\binom{r-3}{s} + \binom{r-3}{s-1}\right]$$
$$+2\left[\binom{r-3}{s-1} + \binom{r-3}{s-2}\right] - \left[\binom{r-3}{s-2} + \binom{r-3}{s-3}\right]$$

$$= \left[\binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2} + 2\binom{r-3}{s-1} - \binom{r-3}{s-3}\right] + \binom{r-3}{s} + \binom{r-3}{s-1}$$
$$+2\binom{r-3}{s-2} - \binom{r-3}{s-2}$$

$$= \left[\binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2} + 2\binom{r-3}{s-1} - \binom{r-3}{s-3}\right] + \left[\binom{r-4}{s} + \binom{r-4}{s-1}\right]$$
$$+\binom{r-3}{s-1} + \binom{r-3}{s-2}$$

$$= \left[\binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2} + 2\binom{r-3}{s-1} + \binom{r-4}{s} - \binom{r-3}{s-3}\right] + \binom{r-4}{s-1}$$
$$+\left\{\binom{r-3}{s-1} + \binom{r-3}{s-2}\right\}$$

$$= \left[\binom{r-4}{s} + 2\binom{r-3}{s-1} + \binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-4}{s-2} - \binom{r-3}{s-3}\right]$$
$$+\left[\binom{r-5}{s-1} + \binom{r-5}{s-2}\right] + \binom{r-2}{s-1}$$

$$= \left[\binom{r-4}{s} + 2\binom{r-3}{s-1} + \binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} + \binom{r-2}{s-1}\right.$$
$$\left.-\binom{r-3}{s-3}\right] + \binom{r-5}{s-2}$$

$$= b_i(G''_{3,k}) + \binom{r-5}{s-2}.$$

This proves (i).

Proof of (ii): If $\frac{k+i}{2} = r$ and $\frac{k-i}{2} = s$, then by (7) and (9), we have,

$$b_i(G''_{3,k}) = \left[\binom{r-4}{s} + 2\binom{r-3}{s-1} + \binom{r-2}{s-2} - \binom{r-2}{s-1} + \binom{r-4}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} - \binom{r-3}{s-3}\right]$$

and

$$b_i(G''_{4,k}) = \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 3\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - \binom{r-4}{s-3}\right].$$

Consider $b_i(G''_{3,k})$

$$= \binom{r-4}{s} + 2\binom{r-3}{s-1} + \binom{r-2}{s-2} + \binom{r-4}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} + \binom{r-2}{s-1} - \binom{r-3}{s-3}$$

$$= \left[\binom{r-5}{s} + \binom{r-5}{s-1}\right] + 2\left[\binom{r-4}{s-1} + \binom{r-4}{s-2}\right] + \left[\binom{r-3}{s-2} + \binom{r-3}{s-3}\right] + \left[\binom{r-3}{s-1} + \binom{r-3}{s-2}\right] + \binom{r-4}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} - \left[\binom{r-4}{s-3} + \binom{r-4}{s-4}\right]$$

$$= \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 2\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} - \binom{r-4}{s-3}\right] + \binom{r-5}{s-1} + 2\binom{r-4}{s-2} + \binom{r-3}{s-3} - \binom{r-4}{s-4}$$

$$= \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 2\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} - \binom{r-4}{s-3}\right] + \left[\binom{r-6}{s-1} + \binom{r-6}{s-2}\right] + 2\binom{r-4}{s-2} + \left[\binom{r-4}{s-3} + \binom{r-4}{s-4}\right] - \binom{r-4}{s-4}$$

$$= \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 2\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} - \binom{r-4}{s-3} + \binom{r-6}{s-1}\right] + \binom{r-6}{s-2} + 2\left[\binom{r-5}{s-2} + \binom{r-5}{s-3}\right] + \binom{r-4}{s-3}$$

$$= \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 2\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - \binom{r-4}{s-3}\right] + \binom{r-6}{s-2} + 2\binom{r-5}{s-3} + \binom{r-4}{s-3}.$$

Now,

$$\binom{r-6}{s-2} + 2\binom{r-5}{s-3} + \binom{r-4}{s-3}$$

$$= \binom{r-6}{s-2} + \left[\binom{r-6}{s-3} + \binom{r-6}{s-4}\right] + \binom{r-5}{s-3} + \binom{r-4}{s-3}$$

$$
\begin{aligned}
&= \left[\binom{r-6}{s-2} + \binom{r-6}{s-3}\right] + \binom{r-6}{s-4} + \binom{r-5}{s-3} + \binom{r-4}{s-3} \\
&= \binom{r-5}{s-2} + \binom{r-6}{s-4} + \binom{r-5}{s-3} + \binom{r-4}{s-3} \\
&= \left[\binom{r-5}{s-2} + \binom{r-5}{s-3}\right] + \binom{r-6}{s-4} + \binom{r-4}{s-3} \\
&= \left[\binom{r-4}{s-2} + \binom{r-4}{s-3}\right] + \binom{r-6}{s-4} \\
&= \binom{r-3}{s-2} + \binom{r-6}{s-4}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
b_i(G''_{3,k}) &= \left[\binom{r-5}{s} + 3\binom{r-4}{s-1} + 2\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-3}{s-2} + \binom{r-5}{s-1}\right. \\
&\quad \left. + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - \binom{r-4}{s-3}\right] + \binom{r-6}{s-4} \\
&= b_i(G''_{4,k}) + \binom{r-6}{s-4}.
\end{aligned}
$$

This proves (ii).

Proof of (iii): If $\frac{k+i}{2} = r$ and $\frac{k-i}{2} = s$, then by (5), we have

$$
b_i(G''_{2,k}) = \left[\binom{r-3}{s} + 2\binom{r-2}{s-1} + \binom{r-3}{s-1} + \binom{r-4}{s-1}\right].
$$

We need to show that $b_i(G''_{4,k}) \geq b_i(G''_{2,k})$. Consider,

$$
\begin{aligned}
&b_i(G''_{4,k}) - b_i(G''_{2,k}) \\
&= \binom{r-5}{s} + 3\binom{r-4}{s-1} + 3\binom{r-3}{s-2} + \binom{r-3}{s-1} + \binom{r-5}{s-1} + \binom{r-4}{s-2} \\
&\quad + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - \binom{r-4}{s-3} - \binom{r-3}{s} - 2\binom{r-2}{s-1} \\
&\quad - \binom{r-3}{s-1} - \binom{r-4}{s-1} \\
&= \left[\binom{r-5}{s} + \binom{r-5}{s-1}\right] + 2\binom{r-4}{s-1} + \left[\binom{r-4}{s-2} + \binom{r-4}{s-3}\right] \\
&\quad + 2\binom{r-3}{s-2} + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} \\
&\quad - \binom{r-4}{s-3} - \binom{r-3}{s} - 2\binom{r-2}{s-1}
\end{aligned}
$$

$$= \binom{r-4}{s} + 2\binom{r-4}{s-1} + \binom{r-4}{s-2} + 2\binom{r-3}{s-2} + \binom{r-4}{s-2} + \binom{r-6}{s-1}$$
$$+2\binom{r-5}{s-2} - \binom{r-3}{s} - 2\binom{r-2}{s-1}$$

$$= \left\{\binom{r-4}{s} + \binom{r-4}{s-1}\right\} + \left\{\binom{r-4}{s-1} + \binom{r-4}{s-2}\right\} + 2\binom{r-3}{s-2}$$
$$+\binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - \binom{r-3}{s} - 2\binom{r-2}{s-1}$$

$$= \binom{r-3}{s} + \binom{r-3}{s-1} + 2\binom{r-3}{s-2} + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2}$$
$$-\binom{r-3}{s} - 2\binom{r-2}{s-1}$$

$$= \left\{\binom{r-3}{s-1} + \binom{r-3}{s-2}\right\} + \binom{r-3}{s-2} + \binom{r-4}{s-2} + \binom{r-6}{s-1}$$
$$+2\binom{r-5}{s-2} - 2\binom{r-2}{s-1}$$

$$= \binom{r-2}{s-1} + \binom{r-3}{s-2} + \binom{r-4}{s-2} + \binom{r-6}{s-1} + 2\binom{r-5}{s-2} - 2\binom{r-2}{s-1}$$

$$= \binom{r-6}{s-1} + 2\binom{r-5}{s-2} + \binom{r-4}{s-2} + \binom{r-3}{s-2} - \binom{r-2}{s-1}$$

$$= \binom{r-6}{s-1} + \left[\binom{r-6}{s-2} + \binom{r-6}{s-3}\right] + \binom{r-5}{s-2} + \binom{r-4}{s-2} + \binom{r-3}{s-2}$$
$$-\binom{r-2}{s-1}$$

$$= \left\{\binom{r-6}{s-1} + \binom{r-6}{s-2}\right\} + \binom{r-6}{s-3} + \binom{r-5}{s-2} + \binom{r-4}{s-2} + \binom{r-3}{s-2}$$
$$-\binom{r-2}{s-1}$$

$$= \left\{\binom{r-5}{s-1} + \binom{r-5}{s-2}\right\} + \binom{r-6}{s-3} + \binom{r-4}{s-2} + \binom{r-3}{s-2} - \binom{r-2}{s-1}$$

$$= \left\{\binom{r-4}{s-1} + \binom{r-4}{s-2}\right\} + \binom{r-6}{s-3} + \binom{r-3}{s-2} - \binom{r-2}{s-1}$$

$$= \left\{\binom{r-3}{s-1} + \binom{r-3}{s-2}\right\} + \binom{r-6}{s-3} - \binom{r-2}{s-1}$$

$$= \left\{\binom{r-2}{s-1}\right\} + \binom{r-6}{s-3} - \binom{r-2}{s-1}$$

$$= \binom{r-6}{s-3} \geq 0.$$

This proves (iii). Hence the theorem.

Corollary 2.6: For $k \geq 10$, we have, $G''_{1,k} \geq G''_{3,k} \geq G''_{4,k} \geq G''_{2,k}$. Consequently,

$$E(G''_{1,k}) \geq E(G''_{3,k}) \geq E(G''_{4,k}) \geq E(G''_{2,k}).$$

Proof: The first statement follows from Theorem 2.5. The second statement follows from Theorem 1.3.

Remark 2.7: The characteristic polynomial and energy of the adjacency matrix of a bipartite graphs $G''_{1,k}$, $G''_{2,k}$, $G''_{3,k}$ and $G''_{4,k}$ for k = 10, 11, 12 (by using *matlab*) are given below:

No. of vertices k	Graphs	Characteristic Polynomial	Energy (approx.)
$k = 10$	$G''_{1,10}$	$\lambda^{10} - 10\lambda^8 + 32\lambda^6 - 36\lambda^4 + 10\lambda^2$	11.7618
	$G''_{3,10}$	$\lambda^{10} - 10\lambda^8 + 31\lambda^6 - 34\lambda^4 + 10\lambda^2$	11.7202
	$G''_{4,10}$	$\lambda^{10} - 10\lambda^8 + 32\lambda^6 - 34\lambda^4 + 9\lambda^2$	11.6828
	$G''_{2,10}$	$\lambda^{10} - 10\lambda^8 + 32\lambda^6 - 33\lambda^4 + 9\lambda^2$	11.6714
$k = 11$	$G''_{1,11}$	$\lambda^{11} - 11\lambda^9 + 41\lambda^7 - 60\lambda^5 + 29\lambda^3 - 2\lambda$	13.1868
	$G''_{3,11}$	$\lambda^{11} - 11\lambda^9 + 40\lambda^7 - 57\lambda^5 + 28\lambda^3 - 2\lambda$	13.1426
	$G''_{4,11}$	$\lambda^{11} - 11\lambda^9 + 40\lambda^7 - 57\lambda^5 + 27\lambda^3 - 2\lambda$	13.1325
	$G''_{2,11}$	$\lambda^{11} - 11\lambda^9 + 40\lambda^7 - 56\lambda^5 + 26\lambda^3 - 2\lambda$	13.1154
$k = 12$	$G''_{1,12}$	$\lambda^{12} - 12\lambda^{10} + 51\lambda^8 - 92\lambda^6 + 65\lambda^4 - 12\lambda^2$	14.3254
	$G''_{3,12}$	$\lambda^{12} - 12\lambda^{10} + 50\lambda^8 - 88\lambda^6 + 62\lambda^4 - 12\lambda^2$	14.2832
	$G''_{4,12}$	$\lambda^{12} - 12\lambda^{10} + 50\lambda^8 - 88\lambda^6 + 61\lambda^4 - 11\lambda^2$	14.2532
	$G''_{2,12}$	$\lambda^{12} - 12\lambda^{10} + 50\lambda^8 - 87\lambda^6 + 59\lambda^4 - 11\lambda^2$	14.2404

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