

# Inequality and Comparison Theory for Non-linear Differential and Integral Equations

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**Abstract :** The present work deals with some basic outcomes related to strict and non-strict non-linear differential and integral inequalities. Further the comparison result is proved and used to discuss the uniqueness theorem.

**Index Terms:** Non-linear differential equations, Strict and Non-Strict Inequalities, Comparison principle, Existence theorem.

## 1. INTRODUCTION

Let  $R$  be real line which is connected set, i.e., which cannot be written in the form of union of two separated sets. Given a bounded interval  $J = [t_0, t_0 + a]$  in  $R$  for some fixed  $t_0, a \in R$  with  $a > 0$ .

Consider the initial value problems for non-linear differential equations

$$\frac{d}{dt}[x(t) - f(t, x(\alpha(t)))] = g(t, x(\alpha(t))), t \in J$$

$$x(t_0) = x_0 \in R \quad \text{and} \quad \alpha(t) \in R \quad (1.1)$$

Where  $f, g : J \times R \rightarrow R$  are continuous real valued functions defined on  $J$ .

By a solution of the (1.1), we mean a function  $x \in C(J, R)$  such that

- i) the function  $t \rightarrow x - f(t, x(\alpha(t)))$  is continuous for each  $x \in R$  and  $\alpha(t)$  be any scalar valued function.
- ii)  $x$  satisfies the equation (1.1).

The importance of the investigations of the work of non-linear differential equations lies in the fact that they include various dynamic systems as special cases.[6,15,16] The consideration of non-linear differential equations is implicit in the work of Krasnoselskii [13] and extensively treated in the various research papers on non-linear differential equations with different perturbations. see Krasnoselskii[13] and references therein. This class of differential and integral equations includes the perturbation of original differential and integral equations in different ways [7,8,9,10,11,12].

In this paper, we initiate the basic theory of non-linear differential equation's of mixed inequalities and existence theorem. Our claim is that, the outcomes of this paper are of basic level and significant contribution to the theory of non-linear ordinary differential equations.

## 2. STRICT AND NON-STRICT INEQUALITIES

We need frequently the following hypothesis in what follows:

(A<sub>0</sub>) The function  $x \rightarrow x - f(t, x(\alpha(t)))$  is increasing in  $R$  for all  $t \in J$ .

We begin by proving the basic results dealing with non-linear differential inequalities.

**Theorem 2.1:** Assume that the hypothesis (A<sub>0</sub>) holds. Suppose that there exist  $y, z \in C(J, R)$

such that 
$$\frac{d}{dt}[y(t) - f(t, y(\alpha(t)))] \leq g[t, y(\alpha(t))], t \in J \quad (2.1)$$

and 
$$\frac{d}{dt}[z(t) - f(t, z(\alpha(t)))] \geq g[t, z(\alpha(t))], t \in J \quad (2.2)$$

If one of the inequalities (2.1) and (2.2) is strict and

$$y(t_0) < z(t_0) \quad (2.3)$$

then 
$$y(t) < z(t) \quad (2.4)$$

for all  $t \in J$ .

**Proof :-** Suppose that the inequality (2.4) is false, then the set  $Z$  defined by

$$Z = \{t \in J : y(t) \geq z(t)\} \quad (2.5)$$

Is non empty.

Denote  $t_1 = \inf Z$  without loss of generality, we may assume that

$$y(t_1) = z(t_1) \quad \text{and} \quad y(t) < z(t) \quad \text{for all } t < t_1.$$

Assume that  $\frac{d}{dt}[z(t) - f(t, z(\alpha(t)))] > g(t, z(\alpha(t)))$  for all  $t \in J$ .

Denote

$$Y(t) = [y(t) - f(t, y(\alpha(t)))] \text{ and}$$

$$Z(t) = [z(t) - f(t, z(\alpha(t)))] \text{ for all } t \in J.$$

Now continuity of  $y$  and  $z$  together with (2.3) implies that there exists a  $t_1 > t_0$  such that

$$y(t_1) = z(t_1) \text{ and } y(t) < z(t) \tag{2.6}$$

For all  $t_0 \leq t \leq t_1$ .

As  $(A_0)$  holds, it follows from (2.5) that

$$Y(t_1) = y(t_1) - f(t_1, y(\alpha(t_1)))$$

$$= z(t_1) - f(t_1, z(\alpha(t_1))) = Z(t_1).$$

and

$$Y(t) = y(t) - f(t, y(\alpha(t)))$$

$$< z(t) - f(t, z(\alpha(t)))$$

$$\Rightarrow Y(t) < Z(t) \tag{2.7} \text{ for all } t_0 \leq t < t_1.$$

From the above relation (2.7), we obtain

$$Y(t_1 + h) = Z(t_1 + h) \text{ and } Y(t_1) < Z(t_1)$$

$$\Rightarrow -Y(t_1) > -Z(t_1)$$

$$\therefore Y(t_1 + h) - Y(t_1) > Z(t_1 + h) - Z(t_1)$$

Dividing both sides by  $h \neq 0$

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

For small  $h < 0$ .

Taking the limit as  $h \rightarrow 0$ , we obtain

$$Y'(t_1) \geq Z'(t_1) \tag{2.8}$$

Hence from inequality (2.7) and (2.8), we get

$$g(t_1, y(\alpha(t_1))) \geq Y'(t_1) \geq Z'(t_1) > g(t_1, z(\alpha(t_1)))$$

Which is a contradiction of our assumption that  $y(t) \geq z(t)$ .

Hence  $y(t) < z(t)$  for all  $t \in J$ .

In the next theorem we discuss the non-strict inequality for the non-linear differential equation (1.1) on  $J$  in which one sided Lipschitz condition used.

**Theorem 2.2 :-** Assume that the hypothesis of theorem 2.1 hold. Suppose also that there exists a real number  $L > 0$  such that  $g(t, y(\alpha(t))) - g(t, z(\alpha(t))) \leq L \text{Sup}_{t_0 < r < t} [y(r) - f(r, y(\alpha(r))) - (z(r) - f(r, z(\alpha(r))))]$  (2.9) Whenever

$y(r) \geq z(r), t_0 \leq r < t$ , then

$$y(t_0) \leq z(t_0) \tag{2.10}$$

$$\Rightarrow y(t) \leq z(t) \tag{2.11}$$

for all  $t \in J$ .

Proof:- Let  $\epsilon > 0$  be given and let  $L > 0$  be any given real number. Set define

$$z_\epsilon(t) - f(t, z_\epsilon(\alpha(t))) = z(t) - f(t, x(\alpha(t))) + e^{2L(t-t_0)} \tag{2.12}$$

so that

$$z_\epsilon(t) - f(t, z_\epsilon(\alpha(t))) > z(t) - f(t, x(\alpha(t)))$$

We define

$$Z_\epsilon(t) = z_\epsilon(t) - f(t, z_\epsilon(\alpha(t))) \text{ and } Z(t) = z(t) - f(t, z(\alpha(t))) \text{ for all } t \in J.$$

Now using inequality (2.9), we have

$$g(t, z_\epsilon(\alpha(t))) - g(t, z(\alpha(t))) \leq L \text{Sup}_{t_0 \leq r < t} [Z_\epsilon(r) - Z(r)]$$

$$= L \epsilon e^{2L(t-t_0)}$$

Now ,

$$\begin{aligned} Z'_\epsilon(t) &= Z(t) + 2L \in e^{2L(t-t_0)} \\ &\geq g(t, z(t)) + 2L \in e^{2L(t-t_0)} \\ &\geq g(t, z_\epsilon(t)) + 2L \in e^{2L(t-t_0)} - L \in e^{2L(t-t_0)} \\ &\Rightarrow Z'_\epsilon(t) \geq g(t, z_\epsilon(t)) \text{ for all } t \in J. \end{aligned}$$

Also we have  $Z_\epsilon(t_0) > Z(t_0) \geq Y(t_0)$ . for all  $t \in J$ .

Now using theorem 2.1 with  $z = z_\epsilon$ , to give

$$Y(t) < Z_\epsilon(t) \text{ for all } t \in J.$$

On taking  $\epsilon \rightarrow 0$  in the above inequality, we get  $Y(t) \leq Z(t)$

Which is further in view of hypothesis  $(A_0)$  implies that (2.11) holds on  $J$ . Hence the proof.

**3. EXISTENCE RESULT**

In this article we prove an existence result for the non-linear differential equation (1.1) On a closed and bounded interval  $J = [t_0, t_0 + a]$  under the mixed Lipschitz and Compactness Conditions on the non-linearity involved in it.

We use the non-linear differential equation (1.1) in the space  $C(J, R)$  of continuous real valued Functions defined on  $[t_0, t_0 + a]$

In  $C(J, R)$  we define a supremum norm  $\| \cdot \|$  as  $\|x\| = \sup_{t \in J} |x(t)|$ . Clearly  $C(J, R)$  is a separable Banach space with respect to the above supremum norm. We prove The existence of solutions for the non-linear differential equation (1.1) via the following fixed point theorem in the Banach spaces.[4]

**Theorem 3.1** Suppose that  $S$  is closed, convex and bounded subset of the separable Banach space  $E$  and let  $A : E \rightarrow E$  and  $B : S \rightarrow E$  be two operators such that

- a)  $A$  is non-linear contraction
- b)  $B$  is compact and continuous, and
- c)  $x = Ax + By$  for all  $y \in S \Rightarrow x \in S$ .

Then the operator equation  $Ax + By = x$  has a solution in  $S$ .

We consider the following hypothesis in what follows.

$$(A_1) \text{ There exists a constant } L > 0 \text{ such that } |f(t, x) - f(t, y)| < \frac{L|x-y|}{M+|x-y|},$$

for all  $t \in J$  and  $x, y \in R$ . moreover  $L \leq M$ .

$$(A_2) \text{ There exists a continuous function } h : J \rightarrow R \text{ such that } |g(t, x)| \leq h(t) \text{ } t \in J, \text{ for all } x \in R.$$

To prove the theorem the following lemma is useful which is discussed in sequel.

**Lemma 3.1** Assume that hypothesis  $(A_0)$  holds. Then for any continuous function  $h : J \rightarrow R$ , the function  $x \in C(J, R)$  is a solution of non-linear differential equation

$$\frac{d}{dt} [x(t) - f(t, x(\alpha(t)))] = h(t) \text{ for all } t \in J, \tag{3.1}$$

$$x(0) = x_0 \in R$$

If and only if  $x$  satisfies the non-linear differential equation

$$x(t) = x_0 - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^t h(s) ds \tag{3.2}$$

Proof :- Let  $h \in C(J, R)$

We first assume that  $x$  satisfy the (3.1) then by definition  $x(t) - f(t, x(\alpha(t)))$  is continuous on the interval  $J = [t_0, t_0 + a]$

and so it is differentiable there, as a result  $\frac{d}{dt} [x(t) - f(t, x(\alpha(t)))]$  is integrable on  $J$ .

Integrating (3.1) from  $t_0$  to  $t$ , we have

$$\int_{t_0}^t \frac{d}{dt} [x(t) - f(t, x(\alpha(t)))] dt = \int_{t_0}^t h(t) dt$$

$$[x(t) - f(t, x(\alpha(t)))]_{t_0}^t = \int_{t_0}^t h(s) ds$$

$$x(t_0) = x_0$$

$$i.e., [x(t) - f(t, x(\alpha(t)))] = [x(t_0) - f(t_0, x(\alpha(t_0)))] + \int_{t_0}^t h(s) ds, t \in J$$

$$\therefore x(t) = x(t_0) - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^t h(s) ds \quad t \in J.$$

Conversely suppose that x satisfies

$$x(t) = x(t_0) - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^t h(s) ds \quad t \in J.$$

Differentiating above equation we get  $\frac{d}{dt}[x(t) - f(t, x(\alpha(t)))] = h(t), t \in J$

Now substituting  $t = t_0$  in (3.2), we get

$$x(t_0) = x_0 - f(t_0, x(\alpha(t_0))) + f(t_0, x(\alpha(t_0)))$$

$$\therefore x(t_0) - f(t_0, x(\alpha(t_0))) = x_0 - f(t_0, x(\alpha(t_0)))$$

Since the mapping  $x \mapsto x - f(t, x)$  is an increasing in  $\mathbb{R}$  for all  $J \in \mathbb{R}$ . Also the mapping  $x \mapsto x - f(t_0, x)$  is one one in  $\mathbb{R}$ . This proves  $x(t_0) = x_0$ . This completes the lemma.

Now we are going to discuss the following existence theorem for the Non-linear differential equation (1.1) on the interval J.

**Theorem 3.2 :** Assume that the hypothesis  $(A_0) - (A_2)$  hold. Then the non-linear differential equation (1.1) has a solution defined on J.

Proof : Let set  $E = C(J, \mathbb{R})$  and define a subset S of E defined by

$$S = \{x \in E \mid \|x\| \leq N\} \tag{3.3}$$

Where  $N = |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \|h\|$

$F_0 > 0$  such that  $F_0 = \sup_{t \in J} |f(t, x(\alpha(t)))|$

Clearly S is a closed, convex and bounded subset of the Banach space E.

Now using the hypothesis  $(A_0)$  and  $(A_2)$  and application of lemma 3.1, we can easily show that the non-linear differential equation (1.1) is equivalent to the non-linear integral equation

$$x(t) = x_0 - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^t g(s, x(\alpha(s))) ds \tag{3.4}$$

for  $t \in J$ .

We define two operators  $A : E \rightarrow E$  and  $B : S \rightarrow E$  by

$$Ax(t) = f(t, x(\alpha(t))), t \in J \tag{3.5}$$

And

$$Bx(t) = x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^t g(s, x(\alpha(s))) ds, t \in J \tag{3.6}$$

Then the integral equation (3.5) is transformed into an operator equation as

$$Ax(t) + Bx(t) = x(t), t \in J \tag{3.7}$$

Our aim is to show that the operators A and B satisfy all the conditions of theorem (3.1).

i.e., we first show that A is a Lipschitz operator on E with the Lipschitz constant L.

Let x, y be any two members in E, then by hypothesis  $(A_1)$

$$|A(x(t)) - A(y(t))| = |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))|$$

$$\leq \frac{L|x(\alpha(t)) - y(\alpha(t))|}{M + |x(\alpha(t)) - y(\alpha(t))|}$$

$$\leq \frac{L\|x - y\|}{M + \|x - y\|} \quad \text{for all } \alpha(t) \in \mathbb{R}, \text{ where } t \in J.$$

This shows that A is a non-linear contraction E with D-function  $\psi$  defined by  $\psi(r) = \frac{Lr}{M + r}$ .

Now we have to show the second condition of theorem (3.1)

i.e., B is compact and continuous operator on S into E.

First we prove, B is continuous on S.

Let  $\{x_n\}$  be a sequence in S converging to a point  $x \in S$ . Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} Bx_n(t) &= \lim_{n \rightarrow \infty} \left[ x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^t g(s, x_n(\alpha(s))) ds \right] \\
&= \lim_{n \rightarrow \infty} x_0 - \lim_{n \rightarrow \infty} f(t_0, x(\alpha(t_0))) + \lim_{n \rightarrow \infty} \int_{t_0}^t g(s, x_n(\alpha(s))) ds \\
&= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^t \left[ \lim_{n \rightarrow \infty} g(s, x_n(\alpha(s))) \right] ds \\
&= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^t g(s, x(\alpha(s))) ds \\
&= Bx(t), \quad \text{for } t \in J.
\end{aligned}$$

Moreover, it can be shown as below that  $\{Bx_n\}$  is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas et.al[16], it is proved that B is a continuous operator on S.

Now we have to show B is compact operator on S.

To prove this it is sufficient to show that B(S) is a uniformly bounded and equicontinuous set in E.

Let  $x \in S$  be arbitrary. Then by hypothesis (A<sub>2</sub>).

$$\begin{aligned}
|Bx(t)| &\leq |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^t |g(s, x(\alpha(s)))| ds \\
&\leq |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^t h(s) ds \\
&\leq |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\|a \quad \text{for all } t \in J.
\end{aligned}$$

Taking supremum over t,

$$\begin{aligned}
\sup_t |Bx(t)| &\leq \sup_t \{ |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\|a \} \\
\|Bx\| &\leq |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\|a, \quad \text{for all } x \in S.
\end{aligned}$$

This shows that B is uniformly bounded on S. Again let  $t_1, t_2 \in J$ . then for any  $x \in S$ , we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left| \int_{t_0}^{t_1} g(s, x(\alpha(s))) ds - \int_{t_0}^{t_2} g(s, x(\alpha(s))) ds \right| \\
&\leq \left| \int_{t_2}^{t_1} g(s, x(\alpha(s))) ds \right| \\
&\leq |p(t_1) - p(t_2)|
\end{aligned}$$

Where  $p(t) = \int_{t_0}^t h(s) ds$ .

Since the function p is continuous on compact J, it is uniformly continuous.

Hence, for  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \epsilon$

For all  $t_1, t_2 \in J$  and for all  $x \in S$ .

This shows that B(S) is an equicontinuous set in E.

Now being uniformly bounded and equicontinuous set in E, so it is compact by Arzela-Ascoli theorem. This proves, B is a continuous and compact operator on S.

Now we have to show that  $x = Ax + By$  for all  $y \in S \Rightarrow x \in S$  is satisfied.

Let  $x \in E$  and  $y \in S$  be arbitrary such that  $x = Ax + By$ .

Then, by assumption (A<sub>1</sub>), we have

$$\begin{aligned}
 |x(t)| &= |Ax(t) + By(t)| \\
 &\leq |Ax(t)| + |By(t)| \\
 &\leq |x_0 - f(t_0, x(\alpha(t_0)))| + |f(t, x(\alpha(t)))| + \int_{t_0}^t |g(s, y(\alpha(s)))| ds \\
 &\leq |x_0 - f(t_0, x(\alpha(t_0)))| + [|f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)|] + \int_{t_0}^t |g(s, y(\alpha(s)))| ds \\
 &\leq |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \int_{t_0}^t h(s) ds
 \end{aligned}$$

Taking supremum over t,

$$\|x\| \leq |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \|h\| a$$

Thus all the conditions of theorem (3.1) are satisfied and hence the operator equation  $Ax + By = x$  has a solution in S. As a result, the non-linear differential equation (1.1) has a solution defined on  $J$ . This completes the proof.

#### 4. MAXIMAL AND MINIMAL SOLUTIONS

Under this section we shall discuss the existence of maximal and minimal solutions for the Non-linear differential equation (1.1) on  $J = [t_0, t_0 + a]$ .

**Definition :** - A solution of the Non-linear differential equation (1.1) is said to be Maximal if for any other solution  $x$  to the Non-linear differential equation (1.1) one has  $x(t) \leq r(t)$ , for all  $t \in J$ . Again, a solution  $\rho$  of the Non-linear differential equation (1.1) is said to be minimal if  $\rho(t) \leq x(t)$ , for all  $t \in J$ , where  $x$  is any solution of the Non-linear differential equation (1.1) existing on  $J$ .

We study the case of Maximal solutions only, as the case of minimal solution is similar and can be proved with the suitable and appropriate modifications.

Given an arbitrary small real number  $\epsilon > 0$ , consider the following IVP of Non-linear differential equation (1.1)

$$\begin{aligned}
 \frac{d}{dt} [x(t) - f(t, x(\alpha(t)))] &= g(t, x(\alpha(t))) + \epsilon, \quad t \in J \quad (4.1) \\
 x(t_0) &= x_0 + \epsilon
 \end{aligned}$$

Where  $f, g \in C(J \times R, R)$ .

An existence theorem for the Non-linear differential equation (1.1) can be stated as follows:

**Theorem 4.1** Assume that the hypothesis  $(A_0)$ - $(A_2)$  hold. Then for every small number  $\epsilon > 0$ , the Non-linear differential equation (1.1) has a solution defined on  $J$ .

Our main existence theorem for maximal solution can be stated as

**Theorem 4.2** Assume that the hypotheses  $(A_0)$ - $(A_2)$  hold. Further  $L \leq M$ , then the Non-linear differential equation (1.1) has a maximal solution defined on  $J$ .

#### 5. COMPARISON THEOREMS

The main use of differential inequalities is to find the bound for the solution set for the differential inequality related to Non-linear differential equation (1.1). In this section we prove that the maximal and minimal solutions serve the bounds for the solutions of the related differential inequality to Non-linear differential equation (1.1) on  $J = [t_0, t_0 + a]$ .

**Theorem 5.1** Assume that the hypotheses  $(A_0)$ - $(A_2)$  hold. Further, if there exists a function  $u \in C(J, R)$  such that

$$\frac{d}{dt} [u(t) - f(t, u(\alpha(t)))] \leq g(t, u(\alpha(t))), \quad t \in J \quad (5.1)$$

$$u(t_0) \leq x_0.$$

Then, 
$$u(t) \leq r(t) \quad (5.2)$$

for all  $t \in J$ , where  $r$  is a maximal solution of the Non-linear differential equation (1.1) on  $J$ .

Proof:- Let  $\epsilon > 0$  be arbitrary small real number. Then by theorem (4.2),  $r(t, \epsilon)$  is a Maximal solution of the non-linear differential equation (4.1) and that the limit

$$r(t) = \lim_{\epsilon \rightarrow 0} r(t, \epsilon) \quad (5.3)$$

is uniform on  $J$  and the function  $r$  is a maximal solution of the non-linear differential equation (1.1) on  $J$ . Hence we obtain

$$\frac{d}{dt} [r(t, \epsilon) - f(t, r(\alpha(t), \epsilon))] = g(t, r(\alpha(t), \epsilon)) + \epsilon, \quad t \in J \quad (5.4)$$

$$r(t_0, \epsilon) = x_0 + \epsilon$$

From above inequality it follows that

$$\frac{d}{dt} [r(t, \epsilon) - f(t, r(\alpha(t), \epsilon))] > g(t, r(\alpha(t), \epsilon)) + \epsilon, t \in J \quad (5.5)$$

$$r(t_0, \epsilon) > x_0$$

Now we apply theorem (2.1) to the inequalities (5.1) and (5.5) and conclude that

$$u(t) < r(t, \epsilon) \quad (5.6)$$

For all  $t \in J$ . This further in view of limit (5.3) implies that inequality (5.2) holds on  $J$ . This completes the proof.

**Theorem 5.2** Assume that the hypotheses (A<sub>0</sub>)-(A<sub>2</sub>) hold. Further, if there exists a function

$v \in C(J, R)$  such that

$$\frac{d}{dt} [v(t) - f(t, v(t))] \geq g(t, v(t)), t \in J \quad (5.7)$$

$$v(t_0) \geq x_0.$$

Then ,

$$\rho(t) \leq v(t) \quad (5.8)$$

for all  $t \in J$ , where  $\rho$  is a minimal solution of the Non-linear differential equation(1.1)on  $J$ .

Note that, Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for the Non-linear differential equation (1.1) on  $J$ . A result in this direction is

**Theorem 5.3 :** Assume that the hypotheses (A<sub>0</sub>)-(A<sub>2</sub>) hold. Suppose that there exists a

Function  $G : J \times R_+ \rightarrow R_+$  such that

$$|g(t, x_1(\alpha(t))) - g(t, x_2(\alpha(t)))| \leq G\left(t, \max_{s \in [t_0, t]} |(x_1(s) - f(s, x_1(\alpha(s)))) - (x_2(s) - f(s, x_2(\alpha(s))))|\right) \quad (5.9)$$

for all  $x_1, x_2 \in E$  and  $x_1, x_2 \in E$ . If identically zero function is the only solution of the differential equation

$$m'(t) = G(t, m(t)), t \in J, m(t_0) = 0, \quad (5.10)$$

then the non-linear differential equation (1.1) has a unique solution defined on  $J$ .

Proof:- By theorem 3.2, the non-linear differential equation (1.1) has a solution on  $J$ .

Suppose that there are two solutions  $u_1$  and  $u_2$  of the non-linear differential equation (1.1)

Existing on  $J$ . Define a function  $m : J \rightarrow R_+$  by

$$m(t) = |(u_1(t) - f(t, u_1(\alpha(t)))) - (u_2(t) - f(t, u_2(\alpha(t))))|. \quad (5.11)$$

As  $(|x(t)|)' \leq |x'(t)|$  for  $t \in J$ , we have that

$$\begin{aligned} m'(t) &\leq \left| \frac{d}{dt} [u_1(t) - f(t, u_1(\alpha(t)))] - \frac{d}{dt} [u_2(t) - f(t, u_2(\alpha(t)))] \right| \\ &\leq |g(t, u_1(t)) - g(t, u_2(t))| \\ &\leq G\left(t, |(u_1(t) - f(t, u_1(\alpha(t)))) - (u_2(t) - f(t, u_2(\alpha(t))))|\right) \\ &= G(t, m(t)) \end{aligned}$$

for all  $t \in J$ ; and that  $m(t_0) = 0$

Now, we apply Theorem 5.1 with  $f \equiv 0$  to get that  $m(t) = 0$  for all  $t \in J$ .

This gives

$$u_1(t) - f(t, u_1(\alpha(t))) = u_2(t) - f(t, u_2(\alpha(t))) \text{ for all } t \in J.$$

Finally, in view of hypothesis (A<sub>0</sub>) we conclude that  $u_1(t) = u_2(t)$  on  $J$ . This completes the proof.

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