RIEMANNIAN MANIFOLDS ADMITTING PSEUDO W_2 – CURVATURE TENSOR $\widetilde{W}$

[1]Suraj Prakash Mishra, [2]Bhagwat Prasad, and [3]Virendra Upadhyaya

[1]Department of Physical Sciences MGCGV, Chitrakoot, Satna, M.P.
[2]Department of Mathematics S.M.M. TOWN PG College, Ballia-277001
[3]Department of Physical Sciences MGCGV, Chitrakoot, Satna, M.P.

Abstract : In this paper we determine some properties of pseudo W_2- curvature tensor denoted by $\widetilde{W}_2$on Riemannian manifolds. Firstly, we consider pseudoW_2- conservative manifolds. After this, spacetime with vanishing pseudo W_2- curvature tensor and some geometrical properties of such a spacetime have been obtained.

Keywords : Energy-Momentum tensor, Einstein field equation, Einstein space, Pseudo W_2-curvature tensor $\widetilde{W}$, Quadratic killing tensor, Quadratic conformal killing tensor.

I. INTRODUCTION

In 2002, Prasad defined and studied a tensor field $\widetilde{P}$ on a Riemannian manifold of dimension (n>2) which includes the projective curvature tensor P. This tensor field $\widetilde{P}$ is known as pseudo-projective curvature tensor and it is given by

$$\widetilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X\text{-}g(X,Z)Y], \qquad (1)$$

where a and b are non-zero constants, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature of the manifold (M^n,g)

If a=1 and $b=-\frac{1}{n-1}$ in (1), then the pseudo-projective curvature tensor takes the form

$$\widetilde{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] = P(X,Y)Z,$$

where P denotes the projective curvature tensor (Mishra, 1984). Thus, the projective curvate tensor P is the particular case of the pseudo-projective curvature tensor $\widetilde{P}$.

Many authors have been investigated pseudo-projective curvature tensor on LP-Sasakian manifold, K-contact manifold, Trans-Sasakian manifold, weakly symmetric manifold and Riemannian manifold [3] [4] [5] [6].

In continuation of the above study, one of the author Maurya in 2004 investigated another curvature tensor on a Riemannian manifold – named as pseudo W_2 – curvature tensor $\widetilde{W}_2$as follows

$$\widetilde{W}_2(X,Y)Z = aR(X,Y)Z + b[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y], \qquad (2)$$

where a and b are non-zero constant, R, S, and r as usual meanings and Q is Ricci operator of the type (1, 1) defined by

$$g(QX,Y)=S(X,Y) \qquad (3)$$

If a=1 and $b==-\frac{1}{n-1}$ then (2) takes the form

$$\widetilde{W}_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY] = W_2(X,Y)Z,$$

where W_2denotes W_2-curvature tensor (Mishra and Pokhariyal, 1970). Thus the W_2- curvature tensor is the particular case of the pseudo-W_2- curvature tensor $\widetilde{W}$.

Prasad [9], Prasad etal [11]and Kumar [10] extended this notation on kenmotsu manifold LP-Sasakian manifold with coefficient α.

After introduction, in section II, some properties of pseudo W_2 – curvature tensor $\widetilde{W}$ are given. In the next section, we study pseudo $\widetilde{W}_2$- conservative manifold. In section III, we characterize a spacetime with vanishing pseudo W_2 - curvature tensor $\widetilde{W}_2$- curvature tensor and some geometrical properties of such a spacetime have been obtained.

II. PROPERTIES OF PSEUDO W_2- CURVATURE TENSOR $\widetilde{W}_2$ ON A RIEMANNIAN MANIFOLD.

Theorem(2.1): Pseudo $\widetilde{W}_2$ – curvature tensor $\widetilde{W}_2$ on a Riemannian manifold satisfied the following algebraic properties:

$$'\widetilde{W}_2(X,Y,Z,W)+'\widetilde{W}_2(Y,X,Z,W)=0 \qquad (2.1)$$

$$'\widetilde{W}_2(X,Y,Z,W)+'\widetilde{W}_2(X,Y,W,Z)=b[g(Y,Z)S(X,W)\text{-}g(X,Z)S(Y,W)+g(Y,W)S(X,Z)\text{-}\ g(X,W)S(Y,Z)]\neq 0 \qquad (2.2)$$

$$'\widetilde{W}_2(X,Y,Z,W)+'\widetilde{W}_2(Z,W,X,Y) \neq 0 \qquad (2.3)$$

and

$$'\widetilde{W}_2\ (X,Y,Z,W)+'\widetilde{W}_2\ (Y,Z,X,W)+'\widetilde{W}_2\ (Z,X,Y,W)=0 \qquad (2.4)$$

$'\widetilde{W}_2(e_i, e_i, Z, W)=0='\widetilde{W}_2(X, Y, e_i, e_i)$ (2.5)
$'\widetilde{W}_2(e_i, Y, Z, e_i)=(a-b)[S(Y-Z)-\frac{r}{n}g(Y,Z)]$ (2.6)
$'\widetilde{W}_2(X, e_i, e_i, W)=b(n-1)[S(X,W)-\frac{r}{n}g(X,W)]-\frac{r}{n}a\, g(X,W)$ (2.7)
$\widetilde{W}_2(e_i, e_i)=0$ (2.8)

Proof : Equation (2) can be written as
$'\widetilde{W}_2(X,Y,Z,W)=a\,'R(X,Y,Z,W)+b[g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]-\frac{r}{n}\left[\frac{a}{n-1}+b\right][g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]$ (2.9)
where
$g(\widetilde{W}_2(X,Y)Z,W)='\widetilde{W}_2(X,Y,Z,W)$
and $g(R(X,Y)Z,W)='R(X,Y,Z,W)$
In view of (2.9);(2.1),(2.2),(2.3) and (2.4) can be proved.
Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1\le i \le n$.
Hence from (2.9), we get (2.5), (2.6), (2.7) and (2.8).
This proves the theorem.

Definition (2.1): Let (M^n,g) be a Riemannian manifold with Levi-Civita connection D.A quadratic killing tensor is a generalization of a killing vector field and is defined as a second order symmetric tensor A satisfying the condition [12] and [14].
$(D_XA)(Y,Z)+(D_YA)(Z,X)+(D_ZA)(X,Y)=0$ (2.10)

Definition(2.2): Let (M^n,g) be a manifold with Levi- Civita connection D. A quadratic conformal killing tensor is an analogous generalization of a conformal killing vector and is defined as a second order symmetric tensor A satisfying the condition [12] and [14].
$(D_XA)(Y,Z)+(D_YA)(Z,X)+(D_ZA)(X,Y)=K(X)g(Y,Z)+K(Y)g(Z,X)+K(Z)g(X,Y)$ (2.11)
Now, we have the following theorem:

Theorem(2.2): If the Ricci tensor of M^n admitting pseudo W_2 – curvature tensor $\widetilde{W}_2$ is a quadratic conformal killing tensor then $\widetilde{W}_2(Y,Z)$ of type (0,2) is also quadratic conformal killing tensor.

Proof: From (2.6), we get
$\widetilde{W}_2(Y,Z)=(a-b)[S(Y,Z)-\frac{r}{n}g(Y,Z)]$, (2.12)
Differentiating covariantly (2.12), we get
$(D_X\widetilde{W}_2)(Y,Z)=(a-b)[(D_XS)(Y,Z)-\frac{1}{n}(D_Xr)g(Y,Z)]$ (2.13)
From (2.13)we get
$(D_X\widetilde{W}_2)(Y,Z)+(D_Y\widetilde{W}_2)(Z,X)+(D_Z\widetilde{W}_2)(X,Y)=(a-b)[\{(D_XS)(Y,Z)+(D_YS)(Z,X)+(D_ZS)(X,Y)\}$
$-\frac{1}{n}\{(D_XS)g(Y,Z)+(D_Yr)g(Z,X)+(D_Zr)g(X,Y)\}]$ (2.14)
If we assume that the Ricci tensor is a quadratic conformal killing tensor, then from (2.11) and (2.14), we get
$(D_X\widetilde{W}_2)(Y,Z)+(D_Y\widetilde{W}_2)(Z,X)+(D_Z\widetilde{W}_2)+(X,Y)=(a-b).[\{K(X)-\frac{1}{n}(D_Xr)\}g(Y,Z)+\{K(Y)-$
$\frac{1}{n}(D_Y r)\}g(Z,X)]+\{K(Z)-\frac{1}{n}(D_Z r)\}g(X,Y)]$ (2.15)
By taking (a-b) K $(X)-\frac{(a-b)}{n}(D_Xr)=\alpha(X)$, then (2.15) can be written as
$(D_X\widetilde{W}_2)(Y,Z)+(D_Y\widetilde{W}_2)(Z,X)+(D_Z\widetilde{W}_2)(X,Y)=[\alpha(X)g(Y,Z)+\alpha(Y)g(Z,X)+\alpha(Z)g(X,Y)]$.
This completes the proof.
Theorem(2.3). Let the Ricci tensor of M^n admitting a pseudo W_2 – curvature tensor $\widetilde{W}_2$ be a quadratic killing tensor. A necessary and sufficient condition for $\widetilde{W}_2(X,Y)$ to be a quadratic killing tensor is that the scalar curvature tensor be constant provided $a-b\neq0$.

Proof: Let us consider that the Ricci tensor of the manifold M^n is a quadratic tensor. Then from equations (2.10) and (2.14), we get
$(D_X\widetilde{W}_2)(Y,Z)+(D_Y\widetilde{W}_2)(Z,X)+(D_Z\widetilde{W}_2)(X,Y)=(a-b)[-\frac{1}{n}\{(D_Xr)g(Y,Z)+(D_Yr)g(Z,X)+(D_Zr)g(X,Y)\}]$ (2.16)
If $\widetilde{W}_2(X,Y)$ is a quadratic killing tensor, then from (2.16), we get
$(a-b)[(D_Xr)g(Y,Z)+(D_Yr)g(Z,X)+(D_Zr)g(X,Y)]=0$ (2.17)
Walkar's Lemma [13] states that, if A(X,Y) and B(X) are numbers such that A(X,Y)=A(Y,X)and
$A(X,Y)B(Z)+A(Y,Z)B(X)+A(Z,X)B(Y)=0$ (2.18)
for all X,Y,Z then either all A(X,Y) are zero or all B(X) are zero.

Hence by above Lemma, we get from (2.17) and (2.18) that either g(X,Y)=0 or $(D_X r)(Y,Z)=0$. But $g(X,Y) \neq 0$ and hence, we get $D_X r=0 \Rightarrow r$ is constant. Conversely, if the scalar curvature is constant, then from (2.16), we get $\widetilde{W}_2$ (X,Y) is a quadratic killing tensor. This proves the theorem.

III. PSEUDO W_2 – CONSERVATIVE RIEMANNIAN MANIFOLDS.

Let$(M^n,g)(n>2)$ be a pseudo W_2 – conservative so div $\widetilde{W}_2 = 0$ [15]. Taking covariant derivative of (2), we get

$$(D_U\widetilde{W}_2)(X,Y)Z=a(D_UR)(X,Y)Z+b[g(Y,Z)(D_UQ)X-g(X,Z)(D_UQ)Y]+\frac{dU(r)}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X-g(X,Z)Y] \tag{2.19}$$

Contracting (2.19) over U, we get

$$(\operatorname{div}\widetilde{W}_2)(X,Y)Z=a(\operatorname{div}R)(X,Y)Z+b[g(Y,Z)(\operatorname{div}Q)X-g(X,Z)(\operatorname{div} Q)(Y)] - \frac{1}{n}[(\frac{a}{n-1}+1)(g(Y,Z))dr(X) - g(X,Z)dr(Y)] \tag{2.20}$$

It is known that $(\operatorname{div} R)(X,Y)Z=(D_XS)(Y,Z)-(D_YS)(X,Z)$ (2.21)

Combining (2.20) and (2.21), we get

$$(\operatorname{div}\widetilde{W}_2)(X,Y)Z = a[(D_XS)(Y,Z)-(D_YS)(X,Z)]+\frac{b(n-1)(n-2)-2a}{2n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{2.22}$$

Thus, in a pseudo W_2 – conservative manifold, relation (2.22) can be put as

$$a[(D_XS)(Y,Z)-(D_YS)(X,Z)]=\frac{2a-b(n-1)(n-2)}{2n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{2.23}$$

Contracting (2.23) with respect to Y and Z, we get $[a-b(n-1)(n-2)]dr(X)=0$ and so either $a-b(n-1)(n-2)=0$ or r=0. Moreover if $a=b(n-1)(n-2)$, then by (2.23), we get

$$(D_XS)(Y,Z) - (D_YS)(X,Z)=\frac{1}{2n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{2.24}$$

From (2.24), we see that div C=0, where C denotes the conformal curvature tensor. Hence in this case the manifold is conformally conservative. On the other hand if $a-b(n-1)(n-2)\neq 0$ then the scalar curvature is constant so from (2.23), we get

$$a[(D_XS)(Y,Z) - (D_YS)(X,Z)]=0 \tag{2.25}$$

Since $a\neq 0$ and hence by (2.25), we get

$(D_XS)(Y,Z)=(D_YS)(X,Z)$,

which means that the manifold has the Codazzi type Ricci tensor [16]. Hence we can state the following theorem:

Theorem(3.1): If $(M^n,g)(n>2)$ be a pseudo W_2 – conservative manifold. Then , either it is conformally conservative or it has constant scalar curvature. Moreover if (M,g)(n>2) is of constant scalar curvature and scalar $a\neq 0$, then Ricci tensor of this manifold is of Codazzi type.

IV. SPACETIME WITH VANISHING PSEUDO W_2 – CURVATURE TENSOR $\widetilde{W}_2$

Let V_4 be the spacetime of general relativity, then from equation (2), we get

$$'\widetilde{W}_2(X,Y,Z,W)=a\,'R(X,Y,Z,W)+b[g(Y,Z)(X,W)-g(X,Z)S(Y,W)] - \frac{r}{4}(\frac{a}{3}+b)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{4.1}$$

If $'\widetilde{W}_2$ (X,Y,Z,W)=0, then (4.1) gives

$$a\,'R(X,Y,Z,W)+b[g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]-\frac{r}{4}(\frac{a}{3}+b)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]=0 \tag{4.2}$$

From (4.2), we get $(a-b)[S(Y,Z) - \frac{r}{4}g(Y,Z)]=0$ (4.3)

Hence, we get state the following theorem:

Theorem(4.1): A pseudo W_2 – flat spacetime is an Einstein spacetime, provided $a-b\neq 0$.

Also from (4.2) and (4.3), we get

$$'R(X,Y,Z,W)=\frac{r}{12}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{4.4}$$

Thus, we have the following theorem:

Theorem(4.2): A pseudo W_2 – flat spacetime is a spacetime of constant curvature, provided $a\neq 0$.

Let us consider a spacetime satisfying the Einstein's field equation with cosmological constant

$$S(X,Y) - \frac{r}{2}g(X,Y)+\lambda g(X,Y)= \kappa T(X,Y) \tag{4.5}$$

where S, r and κ denote the Ricci tensor, scalar curvature and the gravitational constant respectively, λ is the cosmological constant and T(X,Y) is the energy momentum tensor.

Using (4.3) and (4.5), we get

$$T(X,Y)=\frac{1}{\kappa}[\lambda-\frac{r}{4}]g(X,Y) \tag{4.6}$$

Taking covariant derivative of (4.6), we get

$$(D_ZT)(X,Y)=-\frac{1}{4\kappa}dr(Z)g(X,Y) \tag{4.7}$$

Since pseudo W_2 – flat spacetime is constant. Hence

$dr(X)=0$ (4.8)

In view and (4.7) and (4.8), we get $(D_Z T)(X,Y)=0$.

Thus we can state the following theorem:

Theorem(4.3): In a pseudo W_2 – flat spacetime satisfying Einstein's field equation with cosmological constant, the energy momentum tensor is covariant constant.

Katzin etal[17] were the pioneers in caring out a detailed study of curvature collineation(CC), in the context of the related particle and field conservation laws that may be admitted in the standard form of general relativity.

The geometrical symmetrics of a space time are expressed through the equation

$Ł_\xi A - 2\,\Omega\, A = 0,$ (4.9)

where A represents a geometrical/physical quantity, $Ł_\xi$ denotes the Lie derivative with respect to the vector field ξ and Ω is a scalar [17].

One of the most simple and widely used example is the metric inheritance symmetry for A=g in (4.9) and for this case, ξ is the killing vector field if Ω =0. Therefore

$(Ł_\xi\, g)(X,Y) = 2\,\Omega\, g(X,Y)$ (4.10)

A space time M^n is said to admit a symmetry called a curvature collineation (CC)([18][19]), provided there exists a vector field ξ such that

$(Ł_\xi R)(X,Y)Z=0,$ (4.11)

where R is the Riemannian curvature tensor.

Now we shall investigate the role of such symmetry inheritance for the space time admitting pseudo W_2 – curvature tensor $\widetilde{W}_2$.

Let us consider spacetime admitting pseudoW_2 – curvature tensor $\widetilde{W}_2$ with a killing vector field ξ is a CC. Then we have

$(Ł_\xi g)(X,Y)=0,$ (4.12)

Again, since M^n admits a CC, we have from (4.11)

$(Ł_\xi S)(X,Y)=0,$ (4.13)

Taking Lie derivative of (2) and then using (4.11), (4.12) and (4.13), we obtain $(Ł_\xi \widetilde{W}_2)(X,Y)Z=0$.

Thus we have the following theorem:

Theorem(4.4): If a spacetime M^n admitting the pseudo $\mathbf{W_2}$ – curvature tensor $\widetilde{W}_2$with ξ as a killing vector field is CC, then the Lie derivative of the pseudo $\mathbf{W_2}$ – curvature tensor $\widetilde{W}_2$vanishes along the vector field ξ.

The well known symmetry of the energy momentum tensor T is the matter collineation defined by

$(Ł_\xi T)(X,Y)=0,$

where ξ is the vector field generating the symmetry and $Ł_\xi$ is the Lie derivative operator along the vector field ξ.

Let ξ be a killing vector field on the space time with vanishing pseudo $\mathbf{W_2}$ – curvature tensor $\widetilde{W}_2$. Then

$(Ł_\xi g)(X,Y)=0,$ (4.14)

where $Ł_\xi$ denotes Lie derivative with respect to ξ.

Taking Lie derivative of both sides of (4.6) with respect to ξ, we get

$\frac{1}{k}\left(\lambda - \frac{r}{4}\right)'(Ł_\xi\, g)(X,Y)=(Ł_\xi\, T)(X,Y)$ (4.15)

From(4.14) and (4.15), we get

$(Ł_\xi\, T)(X,Y)=0,$

which implies that the spacetime admits matter collineation.

Conversely, if $(Ł_\xi\, T)(X,Y)=0$, it follows from (4.15), we get $(Ł_\xi\, g)(X,Y)=0$.

Hence we can state the following theorem:

Theorem(4.5): If a spacetime obeying Einstein's field equation has vanishing pseudo $\mathbf{W_2}$– curvature tensor $\widetilde{W}_2$then the spacetime admits matter collineation with respect to a vector field ξ if and only if ξ is a killing vector field.

Next, we assume that ξ is a conformal killing vector field. Then we have

$(Ł_\xi\, g)(X,Y) = 2\,\phi\, g(X,Y),$ (4.16)

where ϕ is a scalar.

Then from(4.15), we get

$\left(\lambda - \frac{r}{4}\right).\mathbf{2\,\phi\, g(X,Y)} = \boldsymbol{\kappa}\,(Ł_\xi\, T)(X,Y)$ (4.17)

From (4.6) in (4.17), we find

$(Ł_\xi\, T)(X,Y) = 2\,\phi\, T(X,Y),$ (4.18)

From(4.18), we can say that the energy – momentum tensor has Lie inheritance property along ξ.

Conversely, if (4.18) holds, then it follows that (4.16) holds, that is, ξ is a conformal killing vector field. Thus we have the following theorem.

Theorem(4.6): If a space time obeying Einstein field equation has vanishing pseudo $\mathbf{W_2}$ – curvature tensor $\widetilde{W}_2$, then a vector field ξ on the space time is a conformal killing vector field iff the energy momentum tensor has Lie inheritance along ξ.

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