BOUNDEDNESS OF ITERATIVE COMBINATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

Dr. Manoj Kumar
Faculty Member
Department of Mathematics, R.J. College, Chapra (J.P.U)

Abstract- In this chapter we have studied Boundedness of Iterative Combinations of Bernstein-Kantorovitch Polynomials. Let $f \in L^p[0,1]$ ***, p > 1. The Bernstein - Kantorovitch Polynomials are defined as***

$$K_n(f,x) = (n+1)\sum_{v=0}^{n} n_{C_v} x^v (1-x)^{n-v} \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt$$

$$= (n+1)\sum_{v=0}^{n} p_{n,v}(x) \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt \qquad (1)$$

Again, the iterative combinations $I_{n,k}(f,x)$ ***of operator sequence .*** $\{K_n(f,x)\}_{n\geq 1}$ ***is defined as***

$$I_{n,k}(f,x) = \sum_{m=1}^{k} (-1)^{m+1} k_{C_m} K_n^m(f,x), k \in N,$$

where

$$K_n^2(f,x) = K_n(K_n f,x), K_n^3(f,x) = K_n(K_n^2 f,x), K_n^m(f,x) = K_n(K_n^{m-1} f,x).$$

Here, we show that $I_{n,k}(f,x)$ ***is bounded.***

Keywords :- Iterative combination, Operator sequence, L^p – bounded.

1. INTRODUCTION AND BASIC RESULTS

- Lorent Z[14] defined a sequence of polynomials $\{B_n(f,x)\}_{n>1}$ for $f \in[0,1]$.

by equation (1),

$$\{B_n(f,x)\} = \sum_{v=0}^{n} p_{n,v}(x)(\frac{v}{n}), where$$

$$p_{n,v}(x) = n_{C_v} x^v (1-x)^{n-v}$$

kantorovitch modified equation (1) for $f \in L^p[0,1]$ by

$$K_n(f,x) = (n+1)\sum_{v=0}^{n} p_{n,v}(x) \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt \qquad (2)$$

It cash be also written as

$$K_n(f,x) = \int_0^1 W(n,x,t)f(t)dt, \tag{3}$$

Where ,

$$W(n,x,t) = (n+1)\sum_{v=0}^{n} p_{n,v}(x)\psi_{n,v}(t), \tag{4}$$

$\psi_{n,v}(t)$ is characteristic function of $\left[\frac{v}{n+1}, \frac{v+1}{n+1}\right)$

$K_n(\cdot,x)$ is linear positive operator from $L^p[0,1]$ to $C[0,1]$.

It follows from (2) that

$$K_n(1,x) = 1 \quad ; \; x\epsilon[0,1] \tag{5}$$

$$K_n(t,x) = \frac{2nx+1}{2(n+1)} \tag{6}$$

$$K_n(t^2,x) = \frac{3n(n-1)x^2+6nx+1}{3(n+1)^2} \tag{7}$$

Therefore, first and second order moments are computed as

$$\mu_1(x) = K_n(t-x,x) = \frac{1-2x}{2(n+1)} \tag{8}$$

$$\mu_2(x) = K_n((t-x)^2,x = \frac{3(n-1)\chi+1}{3(n+1)^2} \quad \text{, Where } X = x(x-1) \tag{9}$$

Moreover, the general moment of r^{th} order of Bernstein – Kantorovitch polynomial is related to moments of Bernstein polynomial (LorentZ[14]) by

$$\mu_r(x) = \frac{n+1}{(r+1)\,x(1-x)} \quad B_{n+1}((t-x)^{r+2},x) \tag{10}$$

The iterative combinations $I_{n,k}(f,x)$ of operator sequence $\{K_n(f,x), x\}$n>1 is defined as

$$\text{l}_{n,k}(f,x) = \sum_{n=1}^{k}(-1)^{m+1}k_{C_m}K_n^m(f,x), k \in N, \tag{11}$$

where

$$K_n^2(f,x) = K_n(K_nf,x),$$

$$K_n^m(f,x) = K_n(K_n^{m-1}f,x).$$

2. Boundedness

Theorem 1:- The sequence $\{\mathrm{k_n}(f,.)\}_{n\geq 1}$ is L^p - bounded.

Proof :- we use Holder inequality in summation and then in integration to obtain

$$\left|(n+1)\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)d\,t\right| \leq \sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|d\,t$$

$$\leq \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|dt\right)^{p}\right\}^{\frac{1}{p}} \times$$

$$\times \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\right\}^{\frac{1}{q}}$$

$$\leq \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|^{p}dt\right)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)dt\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$=\left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|^{p}dt\right\}^{\frac{1}{p}}.$$

We next use Fubini's theorem to interchange in

$$\int_0^1 |K_n(f,x)|^p\,dx \leq \sum_{\nu=0}^{n}\int_0^1\int_0^1 (n+1)p_{n,\nu}(x)\,|f(t)|^p\psi_{n,\nu}(t)dtdx$$

$$=\sum_{\nu=0}^{n}\int_0^1 (n+1)\,|f(t)|^p\psi_{n,\nu}(t) \times$$

$$\times\left(\int_0^1 p_{n,\nu}(x)dx\right)dt \qquad (12)$$

$$=\|f\|^p_{L^p[0,1]}.$$

This prove that

$$\|K_n(f,\cdot)\|_{L^p[0,1]} \leq \|f\|_{L^p[0,1]}. \qquad (13)$$

Theorem 2:-

The sequences $\{K_n^m(f,\cdot)\}_{n\geq 1}$ and $\{I_{n,k}(f,\cdot)\}_{n\geq 1}$ are L^p – bounded.

Proof:- We use (13) repeatedly in

$$\begin{aligned} \|K_n^m(f,\cdot)\|_{L^p[0,1]} &= \|K_n\,(K_n^{m-1}f,\cdot)\|_{L^p[0,1]} \\ &\leq \|(K_n^{m-1}f,\cdot)\|_{L^p[0,1]} \\ &\leq \cdots \\ &\leq \|f\|_{L^p[0,1]}, \end{aligned}$$

And using (14)

$$\begin{aligned} \|I_{n,k}(f,\cdot)\|_{L^p[0,1]} &\leq \sum_{m=1}^{k} {}^kC_m\,\|(K_n^m f,\cdot)\|_{L^p[0,1]} \\ &\leq 2^k\|f\|_{L^p[0,1]}\,. \end{aligned}$$

This completes the proof theorem.

3. *Conclusion:* *- Iterative combination of Bernstein Kantorovitch polynomials is L^p-bounded.*

References:

1. *Agrawal, P.N., Singh, Karunesh kumar and Gairola, A.R.: 2010, Lp – Approximation by Iterates of Bernstein Type Polynomial, V.4, No.10, 469-479.*
2. *Agrawal, P.N., Gupta, V. and Gairola, A.R.: 2008, on Iterative Combination of Modified Bernstein- Type Polynomial, Georgian Math.J.15(4), 591-600.*
3. *Berens, H. and Lorentz, G.G.:1972, Inverse Theorem for Bernstein polynomials, Indiana university Maths. J.21, 693-708.*
4. *Butzer,P.L.: 1953 Linear combination of Bernstein Polynomial, Canad. J. Math 5. 559-567.*
5. *Derriennic, M.M.: 1981, Sur L' Approximation De Function Integrable Sur[0,1] Par Des Polymes De Bermtein modifies, J. Approx. Theory 31, 325-343.*
6. *Ditizian, Z. and Ivanov, K.: 1989 Bernstein type Operators and their derivatives, ibid. 56, 72-90.*
7. *Gairola, A.R.: 2009, Approximation by combination of operators of summation- Integral Type, Ph.D. Thesis, Indian institute of technology, Roorkee, India.*
8. *Hrushikesh , N. Mhaskar and Devidas, V. Pai : 2000, Fundamentals of Approximation Theory, Narosa Publishing House, New Delhi.*
9. *Jackson, D.: 1930, The Theory of Appl., Amer. Math. Math. Soc. Coll. Publ. Vol. 11, New York.*
10. *Kasna, H.S. and Agrawal, P.N.:1988, Oh Sharp estimetes and linear combinations of Modified Berstein Polynomials, Bull. Soc. Math. Belg. Ser. B 40(1), 61-71*

11. *Kantorovitch, L.V. : 1930, Sur certains Developments Suivant Les polynomials De La Forme De Bernstein. I.11. C.R. Acad. Sci. URSS, A., 563-568.*
12. *Korokin, P.P.: 1960, Linear Operators and Approximation Theory, Translated from Russian Eddition(1959), Hindustan Publ. Corp. Delhi.*
13. *Kumar, Manoj (2012) : Direct and Inverse Theorems for a class of Linear Operators, Ph.D. Thesis. Magadh University Bodh-Gaya (Bihar).*
14. *Lorentz, G. G. : 1996, Approximation of Functions, Holt Rinehart and Winstom, New-York .*
15. *Upreti, Rita : 1982, On Sequence to Sequence Transformation, Indian J. Pure Appl. Math. 13, 454-457*
16. *Wood, B.: 1984,* L^p*-Approximation of Linear combinations of Integral Bernstein Type operators Anal, Numer. Theo. Approx. 13(1), 65-72.*