ON FUZZY TOTAL CHROMATIC NUMBER OF A TOTALLY REGULAR FUZZY GRAPH

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Abstract: Coloring of fuzzy graphs plays a vital role in theory and practical applications. In this paper, my aim is to explore the practical applications of chromatic number of totally regular fuzzy graph. Here I consider the fuzzy graph with fuzzy set of vertices and fuzzy set of edges.

Keywords: fuzzy graph, totally regular fuzzy graph, fuzzy total chromatic number, chromatic number of totally regular fuzzy graph, Clique, k-clique, Clique number.

1. Introduction

Fuzzy graph theory was first introduced by Rosenfeld [13] in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields. Graph coloring finds its origin during late 1850 and the term "total coloring" was independently introduced by Behzad and Vizing in numerous occasions between 1964 and 1968 [12]. Nagoor Gani and Radha [10] introduced regular fuzzy graph, total degree and totally regular fuzzy graphs. This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy sets by Sattanathan [8] and Parvathi [12]. Edward Samuel and Kayalvizhi [6] introduced chromatic number of totally regular fuzzy graph.

In this paper, focus on fuzzy total chromatic number of a regular fuzzy graph by taking fuzzy set of vertices and fuzzy set of edges. In section 2 review the classical definition of fuzzy graph and other basic definitions of fuzzy graphs. In section 3 fuzzy total chromatic number of a totally regular fuzzy graph and application of totally regular fuzzy graph is presented.

2. Preliminary definitions

Definition 2.1

A *fuzzy graph* $G=(\sigma,\mu)$ is a pair of functions $\sigma: V\to[0,1]$ and $\mu: V\times V\to[0,1]$. Where for all $u,v\in V$, we have $\mu(u,v)\le\sigma(u)\wedge\sigma(v)$.

Definition 2.2

A fuzzy graph is said to be *connected fuzzy graph* if there is at least one path between every pair of vertices in fuzzy graph.

Definition 2.3

Two vertices u and v in G is called *adjacent* if, $\left(\frac{1}{2}\right)\min\{\sigma(u),\sigma(v)\}\le\mu(u,v)$.

Definition 2.4

Two edges v_iv_j and v_jv_k are said to be *incident* if, $2\min\{\mu(v_iv_j),\mu(v_jv_k)\}\le\sigma(v_j)$ for $j=1,2,......,|v|$, $1\le i$, $k\le|v|$.

Definition 2.5

The *degree of vertex* u is $d_G(u) = \sum_{u \neq v} \mu(uv)$. Since $\mu(uv) > 0$ for $uv \in E$ and $\mu(uv) = 0$ for $uv \notin E$.

ie., equivalent to $d_G(u) = \sum_{uv \in E} \mu(uv)$.

Minimum degree of $G = \delta(G) = \wedge\{d(v) \mid v \in V\}$.

Maximum degree of $G = \Delta(G) = \vee\{d(v) \mid v \in V\}$.

Definition 2.6

Let $G = (\sigma, \mu)$ be a fuzzy graph on $G^* = (\sigma, \mu)$. If $d_G(v) = k$ for all $v \in V$. ie., If each vertex has same degree k, then G is said to be a *regular fuzzy graph* of degree k or a k-regular fuzzy graph. Let $G = (\sigma, \mu)$ be a fuzzy graph. Then G is *irregular*, if there is a vertex which is adjacent vertices with distinct degrees.

Definition 2.7

A fuzzy graph $G = (\sigma, \mu)$ is a *complete fuzzy graph*. if, $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. Let $G = (\sigma, \mu)$ be a fuzzy graph.

Definition 2.8

Let G be a graph with vertex set $V(G)$ and edge set $\mathrm{E}(G)$. Vertex coloring of G is a mapping $C : V(G) \to \mathbb{N}$ with $\mathbb{N}$ is a set of natural numbers such that $C(x) \neq C(y)$ if $(x, y) \in \mathrm{E(G)}$. Given an integer k, a coloring of G is a mapping $C : V(G) \to \{1, 2, \ldots\ldots, k\}$ such that $C(x) \neq C(y)$ if $(x, y) \in \mathrm{E(G)}$.

Definition 2.9

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ of fuzzy sets on V is called a k-*fuzzy coloring* of $G = (V, \sigma, \mu)$ if,

(a) $V\Gamma = \sigma$,

(b) $\gamma_i \wedge \gamma_j = 0$,

(c) For every strong edge xy of G, $\min\{\gamma_i(x), \gamma_i(y)\} = 0$, $(1 \leq i \leq k)$.

Definition 2.10

The minimum number k for which there exists a k- fuzzy coloring is called the *fuzzy chromatic number* of G, denoted as $\chi^f(G)$. The chromatic number of fuzzy graph $G : (V, \sigma, \mu)$ is defined as $\chi(G) = \max\{X_\alpha \mid \alpha \in L\}$ where $\chi_\alpha = \chi(G_\alpha)$.

Definition 2.11

Let $G = (\sigma, \mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u \in v$ is defined by,

$$td_G(v) = \sum_{u \neq v} \mu(uv) + \sigma(u) = \sum_{uv = E} \mu(uv) + \sigma(u) = d_G(u) + \sigma(u).$$

If each vertex of G has the same total degree k, then G is said to be a *totally regular fuzzy graph* of total degree k or a k-*totally regular fuzzy graph*.

Definition 2.12

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a k-fuzzy total coloring of $G = (V, \sigma, \mu)$ if,

(a) $\max_i \gamma_i(v) = \sigma(v)$ for all $v \in V$ and
$\max_i \gamma_i(uv) = \mu(uv)$ for all edge $uv \in E$,

(b) $\gamma_i \wedge \gamma_j = 0$,

(c) For every adjacent vertices u, v of $\min\{\gamma_i(u), \gamma_i(v)\} = 0$ and for every incident edges $\min\{\gamma_i(v_j v_k \mid v_j v_k$ are set of incident edges from the vertex v_j, $j = 1, 2, \ldots\ldots, |v|\}$.

The least value of k for which G has a k-fuzzy total coloring, denoted by $\chi_T(G)$ is called the *fuzzy total chromatic number* of G.

Definition 2.13

A Complete fuzzy subgraph of G is also called a clique of G. A clique of order k is a k-clique. The maximum order of a clique of G is called the clique number of G and is denoted by $\omega(G)$. Thus $\alpha(G) = \omega(G)$ for every graph G.

Definition 2.14 [6]

Let $G = (\sigma, \mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u \in v$ is defined by,

$$td_{CG}(v) = \sum_{u \neq v} \mu_C(uv) + \sigma_C(u)$$

$$= \sum_{uv = E} \mu_C(uv) + \sigma_C(u)$$

$$= d_{CG}(u) + \sigma_C(u)$$

If each vertex of G has the same total degree k, then G is said to be a *Chromatic number of totally regular fuzzy graph* of total degree k or a k *- chromatic number of totally regular fuzzy graph*.

3. Fuzzy total chromatic number of a totally regular fuzzy graph

Consider the following example of fuzzy total coloring of a totally regular fuzzy graph and its associated total chromatic number a scheduling problem is presented in example.

Example:

In a rural community, there are Ten Children's, which we denoted by $T = \{c_1, c_2, \ldots\ldots, c_{10}\}$, living in ten different homes who require physical therapy sessions during the week. Ten physical therapists in a neighboring city have volunteered to visit the same day. The set of children visited by a physical therapist on any one day is referred to as a tour. It is decided that an optimal number of children to visit on a tour is 4. The following ten tours are agreed upon:

$$T_1 = \{c_1, c_2, c_3, c_4\}, \quad T_2 = \{c_3, c_5, c_7, c_9\},$$

$$T_3 = \{c_1, c_2, c_9, c_{10}\}, \quad T_4 = \{c_4, c_6, c_7, c_8\},$$

$$T_5 = \{c_2, c_5, c_9, c_{10}\}, \quad T_6 = \{c_1, c_4, c_6, c_8\},$$

$$T_7 = \{c_3, c_4, c_8, c_9\}, \quad T_8 = \{c_2, c_5, c_7, c_{10}\},$$

$$T_9 = \{c_5, c_6, c_8, c_{10}\}, \quad T_{10} = \{c_6, c_7, c_8, c_9\}.$$

It would be preferred if all ten tours can take place during Monday through Friday but the physical therapists are willing to work on the weekend if necessary. Is it necessary for someone to work on the weekend? [4]

Solution

A graph G is constructed with vertex set is $V(G) = \{T_1, T_2, ..., T_{10}\}$, where T_i is adjacent to T_j $(i \neq j)$ if $T_i \cap T_j \neq 0$ (see figure 1.1).

The minimum number of days needed for these tours is $\chi(G)$.

Since $\{T_2, T_3, T_5, T_7, T_9, T_{10}\}$ induces a maximum clique in G. It follows that $\omega(G) = 6$.

By theorem, "For every graph G, $\chi(G) \geq \omega(G)$"

$$\omega(G) \geq 6$$

There is a 6-coloring of G (see figure 1.1) and so $\chi(G) = 6$. Thus, visiting all ten children's requires 6 days and it is necessary for some physical therapist to work on the weekend.

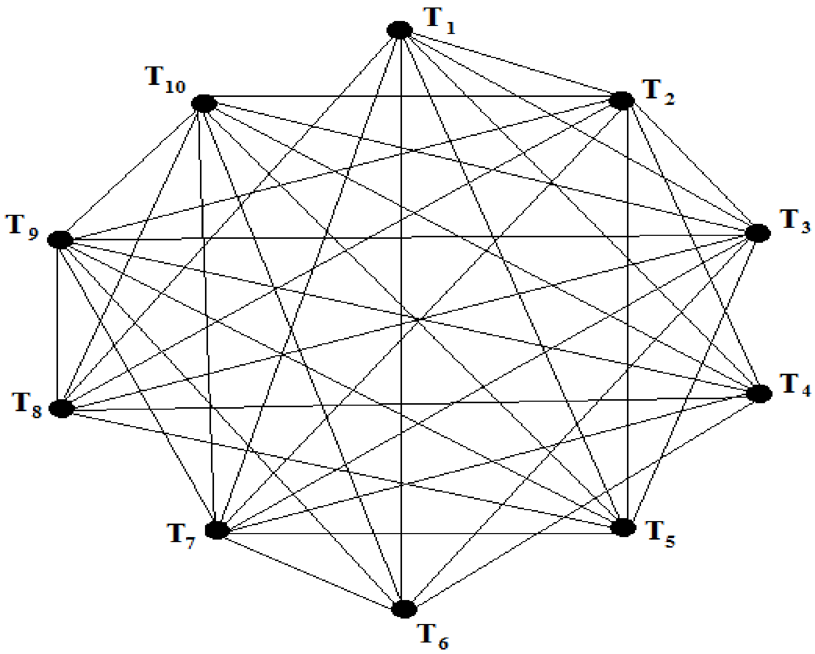

Figure 1.1

Consider the figure (2.2) a totally regular fuzzy graph $G_k = (V, \sigma, \mu)$ but not regular crisp graph and regular fuzzy graph with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and edge set $E = \{v_i v_j / ij = 12, 15, 16, 17, 23, 26, 27, 34, 36, 37, 45, 46, 47, 56, 57\}$ the membership functions are defined as follows.

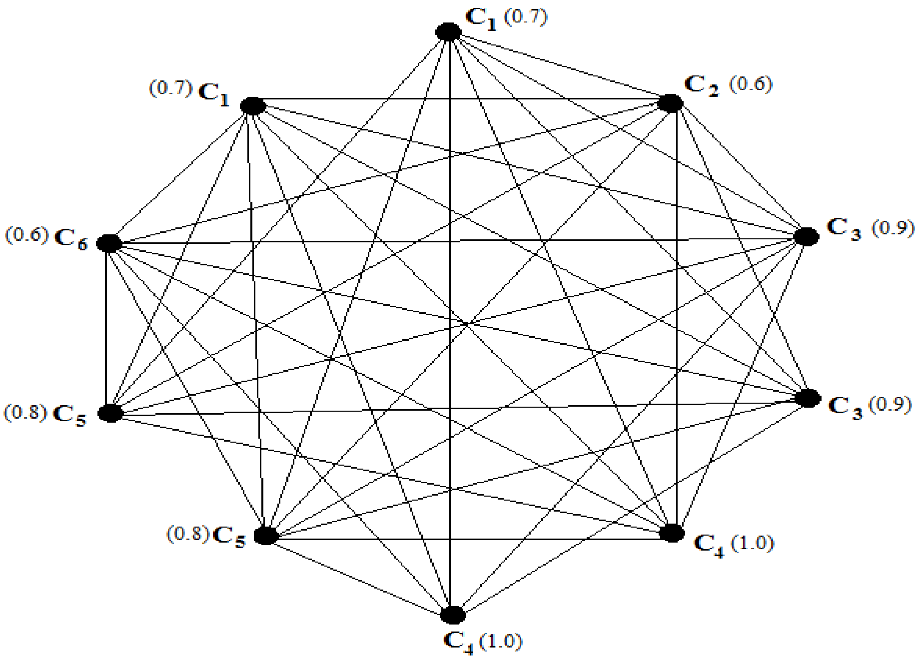

Figure 1.2

$$\sigma(v_i)=\begin{cases}1.0 & for \quad i=5,6\\ 0.9 & for \quad i=3,4\\ 0.8 & for \quad i=7,8\\ 0.7 & for \quad i=1,10\\ 0.6 & for \quad i=2,9\end{cases};$$

$$\mu(v_iv_j)=\begin{cases}0.25 & for \quad ij=13,17,25,610,89,810\\ 0.20 & for \quad ij=12,15,24,28,39,46,69,710\\ 0.15 & for \quad ij=14,27,29,38,47,49,510\\ 0.10 & for \quad ij=16,210,35,37,310,48,58,67,79,910\\ 0.05 & for \quad ij=18,23,36,410,57,59\end{cases}$$

We see that the membership functions satisfy the definition of totally regular fuzzy graph. Let $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\}$ be a family of fuzzy sets defined on $V\cup E$ as follows.

$$\gamma_1(v_i)=\begin{cases}0.7, & i=1,10\\ 0, & otherwise\end{cases} \quad ; \quad \gamma_1(v_iv_j)=\begin{cases}0.25, & ij=13,17\\ 0.20, & ij=12,15\\ 0.15, & ij=14\\ 0.10, & ij=16\\ 0.05, & ij=18\\ 0, & otherwise\end{cases}$$

$$\gamma_2(v_i)=\begin{cases}0.6, & i=2\\ 0, & otherwise\end{cases} \quad ; \quad \gamma_2(v_iv_j)=\begin{cases}0.25, & ij=25\\ 0.20, & ij=24,28\\ 0.15, & ij=27,29\\ 0.10, & ij=210\\ 0.05, & ij=23\\ 0, & otherwise\end{cases}$$

$$\gamma_3(v_i)=\begin{cases}0.9\ , & i=3,4\\ 0\ , & otherwise\end{cases} \quad ; \quad \gamma_3(v_iv_j)=\begin{cases}0.20\ , & ij=39,46\\ 0.15\ , & ij=38,47,49\\ 0.10\ , & ij=35,37,310,48\\ 0.05\ , & ij=36,410\\ 0\ , & otherwise\end{cases}$$

$$\gamma_4(v_i)=\begin{cases}1.0\ , & i=5,6\\ 0\ , & otherwise\end{cases} \quad ; \quad \gamma_4(v_iv_j)=\begin{cases}0.25\ , & ij=610\\ 0.20\ , & ij=69\\ 0.15\ , & ij=510\\ 0.10\ , & ij=58,67\\ 0.05\ , & ij=57,59\\ 0\ , & otherwise\end{cases}$$

$$\gamma_5(v_i)=\begin{cases}0.8\ , & i=7,8\\ 0\ , & otherwise\end{cases} \quad ; \quad \gamma_5(v_iv_j)=\begin{cases}0.20\ , & ij=710\\ 0.10\ , & ij=79\\ 0\ , & otherwise\end{cases}$$

$$\gamma_6(v_i)=\begin{cases}0.6\ , & i=6\\ 0\ , & otherwise\end{cases} \quad ; \quad \gamma_6(v_iv_j)=\begin{cases}0.10\ , & ij=910\\ 0\ , & otherwise\end{cases}$$

Hence the family $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\}$ satisfies our definitions of total coloring of a totally regular fuzzy graph. We defined that any family of fuzzy sets having less than 6 times periods could not satisfy the definition. From the tables given below, we can see the values of $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ clearly.

Hence in this case totally regular fuzzy graph of total chromatic number $\chi_T(G_k)$ is 6.

Table 1

Sets of vertices (v_i)

Vertices	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	**Max**
1	0.7	0	0	0	0	0	**0.7**
2	0	0.6	0	0	0	0	**0.6**
3	0	0	0.9	0	0	0	**0.9**
4	0	0	0.9	0	0	0	**0.9**
5	0	0	0	1.0	0	0	**1.0**
6	0	0	0	1.0	0	0	**1.0**
7	0	0	0	0	0.8	0	**0.8**
8	0	0	0	0	0.8	0	**0.8**
9	0	0	0	0	0	0.6	**0.6**
10	0.7	0	0	0	0	0	**0.7**

Table 2

Sets of Edges $(v_i v_j)$

Edges	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	Max
12	0.20	0	0	0	0	0	**0.20**
13	0.25	0	0	0	0	0	**0.25**
14	0.15	0	0	0	0	0	**0.15**
15	0.20	0	0	0	0	0	**0.20**
16	0.10	0	0	0	0	0	**0.10**
17	0.25	0	0	0	0	0	**0.25**
18	0.05	0	0	0	0	0	**0.05**
23	0	0.05	0	0	0	0	**0.05**
24	0	0.20	0	0	0	0	**0.20**
25	0	0.25	0	0	0	0	**0.25**
27	0	0.15	0	0	0	0	**0.15**
28	0	0.20	0	0	0	0	**0.20**
29	0	0.15	0	0	0	0	**0.15**
210	0	0.10	0	0	0	0	**0.10**
35	0	0	0.10	0	0	0	**0.10**
36	0	0	0.05	0	0	0	**0.05**
37	0	0	0.10	0	0	0	**0.10**
38	0	0	0.15	0	0	0	**0.15**
39	0	0	0.20	0	0	0	**0.20**
310	0	0	0.10	0	0	0	**0.10**
46	0	0	0.20	0	0	0	**0.20**
47	0	0	0.15	0	0	0	**0.15**
48	0	0	0.10	0	0	0	**0.10**
49	0	0	0.15	0	0	0	**0.15**
410	0	0	0.05	0	0	0	**0.05**
57	0	0	0	0.05	0	0	**0.05**
58	0	0	0	0.10	0	0	**0.10**
59	0	0	0	0.05	0	0	**0.05**
510	0	0	0	0.15	0	0	**0.15**
67	0	0	0	0.10	0	0	**0.10**
69	0	0	0	0.20	0	0	**0.20**
610	0	0	0	0.25	0	0	**0.25**
79	0	0	0	0	0.10	0	**0.10**
710	0	0	0	0	0.20	0	**0.20**
89	0	0	0	0	0	0.25	**0.25**
810	0	0	0	0	0	0.25	**0.25**
910	0.10	0	0	0	0	0	**0.10**

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