APPLICATIONS OF PICARD ISHIKAWA TYPE FIXED POINT ITERATION FOR CONSTRUCTION OF FRACTALS

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Abstract: Complex Polynomials $P_{\mathbb{C}}$(x) = x^n + m x + r where m, r ∈ ℂ appeared in various engineering problems such as digital image processing. These Polynomials are very useful for the determination of pole-zero plots for signals and to study the structure and solutions of linear time invariant space models [15], hence the study of behaviour of these polynomials and their Julia and Mandelbrot sets is interested area of research. Some researchers introduced Julia and Mandelbrot sets in implicit Jungck Mann and Jungck Ishikawa orbits and they worked on this implicit iteration process to construct graphical structure of these complex polynomials, for this they divide the polynomial into two parts as P_1(x) = x^n + r and P_2(x) = m x then the Jungck iteration process and its variants are useful for determination of the common fixed points of these two maps. So we get an escape criterion and generate fractals for polynomials of mentioned form with the help of iteration processes. In present paper we apply Picard Ishikawa type fixed point iteration method for the construction of fractals such as Mandelbrot set and Julia set for some complex polynomials and obtain graphical output of these sets, we establishes some results for the construction of Mandelbrot set and Julia set with time escape. Also we compare some attributes or parameters with the help of advanced software's like MATLAB. In particularly the graphical behaviour of the complex polynomial of the form $P_{\mathbb{C}}$(x) = x^n + m x + r where m, r ∈ ℂ and n ≥ 2 by using Picard Ishikawa type fixed point iteration process for the construction of fractals.

Index Terms: Fixed point, iteration, fractal geometry, MATLAB.

1. Introduction

Fixed point theory gives us appropriate tools for the understanding and study of so many nonlinear phenomena occurring in the field of sciences, geometry, biology and complex graphics [1, 2, 3]. Complex graphics structures like fractals represents fixed point of some set maps [1]. Fractals are the nonlinear phenomena of unfolding symmetries or expanding symmetries of self-similar patterns representing on any scale. Fractals are the self-similar mathematical structures which represents the symmetry and similarity for considerably small portions of the object to the entire object. Fractals have irregular geometric structure not like to that of Euclidean geometry. Julia [4] is one of the pioneers of fractal geometry, who studied iterated complex polynomials and brings a new set known as Julia set which is a classical example of fractals. Consider ℂ as the complex space, let P: ℂ → ℂ be a complex polynomial of degree n ≥ 2 with complex coefficients and $\mathrm{P}^{\mathrm{i}}(x)$ be the i^{th} iterate of x, for large values of i this iteration process gives Julia set [1, 5, 6, 7, 8].

Definition 1 [1]

The set of points in ℂ whose orbits do not converge to a point at infinity is known as Filled Julia set, $\mathrm{F_P}$ and is

$$\mathrm{F_P} = \{x \in \mathbb{C} / \left|\mathrm{P}^{\mathrm{i}}(x)\right| \; is\ bounded\ for\ i = 0, \dots, \infty\}$$

Julia set of P is denoted by $\mathrm{J_P}$it is the boundary of Filled Julia set, hence $\mathrm{J_P} = \delta \mathrm{F_P}$.

Therefore, as $x \in \mathrm{J_P}$ if for every neighbourhood of x there exist points w and v such that $\mathrm{P}^{\mathrm{i}}(w) \to \infty$ and $\mathrm{P}^{\mathrm{i}}(v) \nrightarrow \infty$.

The complement of a Julia set is a Fatou set.

Let p ∈ ℂ be a fixed point of P and $\left|\mathrm{P}^{i'}(p)\right|$ = p, a point p is called a periodic point if p = P^{i}p for some integer i ≥ 0.

Let {p, Pp, . . ., P^{i}p, . . .} be an orbit of p. Also the point p is called as an attracting point if 0 ≤ p < 1and a repelling if p >1 [5, 6].

The following results gives a significant relation between repelling points of a polynomial and the Julia set.

2. Picard Ishikawa type Fixed Point Iteration.

Theorem 1 [5]

If P is a complex polynomial, then $\mathrm{J_P}$ is the closure of the repelling periodic points of P.

Definition 2.1

If p is an attracting fixed point of P, then the set A(p) is known as the basin of attraction of p if

A(p) = { $x \in \mathbb{C}$ / $\mathrm{P}^{\mathrm{i}}(x) \to p\ as\ i \to \infty$}

Also basin of attraction of infinity is defined as A(∞) = { $x \in \mathbb{C}$ / $\mathrm{P}^{\mathrm{i}}(x) \to \infty\ as\ i \to \infty$}

Lemma 1[6]

Let p be an attracting fixed point of P then $\mathrm{J_P} = \delta$A(p)

This Lemma 1 is useful for construction of Julia sets and this shows that the Julia set represent as the boundary of the basin of attraction of each attracting fixed point of P with ∞. Fixed point p for any complex polynomial exist due to the Brouwer fixed point theorem [18].

The existence of an attracting fixed point depends on the choice of the parameters. Consider the polynomial P_r(x) = x^2 + r, it has two fixed points in which one is ∞ in this case a fixed point p is attracting if $|2p|< 1$ hence Fix(v_r) = $\left(\frac{1}{4} - r\right)^{\frac{1}{2}}$ then the Julia set J_{P_r} on the real axis where r = 0 are reflection symmetric while those with complex parameter values r ∈ ℂ indicates rotational symmetry.

Fractals are introduced by Mandelbrot [8] which is extension of Julia set [4] he searched the graphical behaviour of connected Julia sets and plotted them for complex functions, P_r(x) = x^2 + r, where x, r ∈ ℂ, x is complex variable and r is input parameter. Also he identified various geometrical properties with dimensions, symmetry and similarity etc. which is very important in fractal geometry.

Definition 2.2 [6]

Let P be any complex polynomial of degree n ≥ 2. A Mandelbrot set M is the set consisting of all parameters r for which the Julia set, J_{P_r} is connected, that is, M = {r ∈ ℂ/ J_{P_r} $is\ connected\}$ or an equivalently M = {r ∈ ℂ / $\{|P_r^n(0)|\} \nrightarrow \infty\ as\ n \rightarrow \infty\}$.

Mandelbrot [8] found that records of heart beat, variations of traffic flow, irregular coastal structures, and so many naturally existing textures are representing fractals.

For the construction of fractals various procedures are applied such as iterated functions system, random fractals, escape time criterion etc. The escape time criterion is the algorithmic stopping procedure which is based on number of iterations necessary to determine that orbit sequence tends to infinity or not. This algorithm is useful to demonstrate some attributes of dynamic system in iteration process.

In general the escape criterion for Julia and Mandelbrot sets is as follows

Theorem 2 [5]

For P_r(x) = x^2 + r, where x, r ∈ ℂ, if there exists $i \geq 0$ such that $|P_r^i(x)|$ > max$\{|r|, 2\}$ then $P_r^i(x) \rightarrow \infty$ as $i \rightarrow \infty$, where the term max$\{|r|, 2\}$ is escape radius threshold.

In each iteration the escape radius varies and it is very useful in in the constructions of fractals.

Initially Julia and Mandelbrot sets are study for the polynomials P_r(x) = x^2 + r, where x, r ∈ ℂ then furthermore it is generalized for 3rd and nth degree complex polynomials. The Julia set for nth degree complex polynomial P_r(x) = x^n + r, n ∈ ℕ first analysed by Lakhatakia et.al. [19].

So many researchers utilized different iterative processes for the construction of fractals. Julia and Mandelbrot sets are generally studied for 2nd, 3rd and nth degree polynomials in Picard orbit [20].

Let P: ℂ → ℂ with x_0 ∈ ℂ the Picard orbit [5] is a sequence $\{x_i\}$ defined as x_{i+1} = P(x_i), for $i \geq 0$ but the slow convergence is the drawback of Picard iteration process so various following faster converging iteration process are introduced by researchers to construct Julia and Mandelbrot sets.

One-step Mann iteration process to construct superior Julia and Mandelbrot sets for nth degree complex polynomials P_r(x) = x^n + r, first used by Rani and Kumar [21, 22] in which the Mann orbit, for any x_0 ∈ ℂ, is a sequence $\{x_i\}$ defined as x_{i+1} = $(1 - \alpha)x_i + \alpha P(x_i)$, for $i \geq 0$ and α ∈ (0, 1].

A two-step Ishikawa iteration process used by Rani and Chauhan [10] and Chauhan et. al. [11] in the year 2010 for the study of relative superior Julia and relative superior Mandelbrot sets, respectively. The dynamics of the nth order complex polynomial for non-integer values were searched [12] also they construct new Julia and Mandelbrot sets using Ishikawa orbit, for any x_0 ∈ ℂ is a sequence $\{x_i\}$ defined as

$$\begin{cases} x_{i+1} = (1 - \alpha)x_i + \alpha P y_i, \\ y_i = (1 - \beta)x_i + \beta P x_i, \end{cases}$$

where, $i = 0, \dots, \infty$ and α, β ∈ (0, 1].

The three-step Noor iteration process for Julia and Mandelbrot sets search out by Rani and Ashish [23], in the Noor orbit for any x_0 ∈ ℂ is a sequence $\{x_i\}$ defined as

$$\begin{cases} x_{i+1} = (1 - \alpha)P x_i + \alpha_i P y_i, \\ y_i = (1 - \beta)P x_i + \alpha P u_i, \\ u_i = (1 - \gamma)P x_i + \gamma P x_i, \end{cases}$$

where, $i = 0, \dots, \infty$ and α, β, γ ∈ (0, 1].

Kang et. al. [13, 24] worked with the modified Ishikawa iteration process, called as S-iteration for the study of superior Mandelbrot sets, tricorns and multicorns. The S-Orbit for any x_0 ∈ ℂ is a sequence $\{x_i\}$ defined as

$$\begin{cases} x_{i+1} = (1 - \alpha)x_i + \alpha P y_i, \\ y_i = (1 - \beta)x_i + \alpha P x_i, \end{cases}$$

where, $i = 0, \dots, \infty$ and α, β ∈ (0, 1].

A four-step iteration process which has faster convergence than Picard, Mann and S-iteration processes used by Kumari et. al. [14] and get some generalizations of Julia and Mandelbrot sets for 2nd, 3rd and nth degree complex polynomials.

The Picard Ishikawa type fixed point iteration process introduced by Piri et al. [17] this iteration process converges faster than Mann and Ishikawa iteration processes.

Let B be a subset of a Banach space and P: B →B then the three step iteration process is given by

$$\begin{cases} x_1 = x \in B, \\ x_{i+1} = (1-\alpha_i)y_i + \alpha_i P y_i, \\ y_i = P z_i, \\ t_i = (1-\beta_i)x_i + \beta_i P x_i, \end{cases}$$

here, $i \geq 0$ and $\alpha_i, \beta_i \in (0, 1]$.

3. Complex Polynomials in a Picard Ishikawa Type Orbit with Escape Criterion

Here with the use of a Picard Ishikawa iterative process some escape criterions are proved for the determination of escape radius. In all complex polynomials the parameters are selected in such a way that there exist at least one attracting fixed point.

Let $\mathbb{C}$ be a complex space and $P_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$ be a complex polynomial with complex coefficients. The Picard Ishikawa orbit around any $x_0 \in \mathbb{C}$, is a sequence $\{x_i\}$ defined by

$$\begin{cases} x_{i+1} = (1-\alpha)y_i + \alpha P_{\mathbb{C}} y_i, \\ y_i = P_{\mathbb{C}} z_i, \\ z_i = P_{\mathbb{C}} t_i, \\ t_i = (1-\beta)x_i + \beta P_{\mathbb{C}} x_i, \end{cases}$$

here, $i = 0, \ldots, \infty$ and $\alpha, \beta \in (0, 1]$.

3.1 n^th Degree Complex Polynomials in a Picard Ishikawa Type Orbit with Escape Criterion

Now for n^th Degree complex polynomial $P_{\mathbb{C}}(x) = x^n + m\,x + r$ where m, r $\in \mathbb{C}$, the following result gives escape criterion in a Picard Ishikawa type orbit.

Theorem 3.1

Suppose $|x| \geq |r| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}\right\}$, with n ≥ 2 and $\alpha, \beta \in (0, 1]$. Define a sequence $\{x_i\}$, i ∈ ℕ as

$$\begin{cases} x_{i+1} = (1-\alpha)y_i + \alpha P_{\mathbb{C}} y_i, \\ y_i = P_{\mathbb{C}} z_i, \\ z_i = P_{\mathbb{C}} t_i, \\ t_i = (1-\beta)x_i + \beta P_{\mathbb{C}} x_i, \end{cases}$$

where x_0 = x, y_0 = y, z_0 = z, and t_0 = t. Then $|x_i| \to \infty$ as $i \to \infty$.

Proof: Let $P_{\mathbb{C}}(x) = x^n + m\,x + r$ and $|x| \geq |r|$, $\beta \leq 1$ with defined sequence $\{x_i\}$ i ∈ ℕ gives

$$\begin{aligned} |t| &= |(1-\beta)x + \beta P_{\mathbb{C}}(x)| \\ &\geq |(1-\beta)x + \beta(x^n + m\,x + r)| \\ &\geq |(1-\beta)x + \beta(x^n + m\,x)| - \beta|r| \\ &\geq \beta|x^n| - (1 - \beta + \beta|m|)\,|x| - \beta|x| \\ &\geq |x|\big(\beta|x^{n-1}| - (1 + \beta|m|)\big) \\ &\geq |x|\big(\beta|x^{n-1}| - (1 + |m|)\big) \end{aligned}$$

Hence, $|t| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$ by assumption, $|x| > \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}$ and hence $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > 1$ this implies that $|t| > |x|$.

Since z_0 = z, then defined sequence $\{x_i\}$, i ∈ ℕ gives $|z| \geq |t^n + mt| - |r|$.

As $\beta \leq 1$ and $|x| \geq |r|$, gives $|z| \geq |t|\left(\frac{\beta|t^{n-1}|}{1+|m|} - 1\right)$ and hence $\left(\frac{\beta|t^{n-1}|}{1+|m|} - 1\right) > 1$ this implies that $|z| > |x|$.

As y_0 = y, $|y| = |P_{\mathbb{C}}(z)| = |z^n + mz + r|$, so proceeding as earlier way gives $|y| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > |x|$,

from $|x| \geq |r|$, and $\alpha \leq 1$ gives

$$\begin{aligned} |x_1| = |(1-\alpha)y + \alpha P_{\mathbb{C}}(y)| &= |(1-\alpha)y + \alpha(y^n + my + r)| \\ &\geq |(1-\alpha)y + \alpha(y^n + my)| - \alpha|r| \\ &\geq \alpha|y^n| - (1 - \alpha + \alpha|m|)\,|y| - \alpha|y| \\ &\geq |y|\big(\alpha|y^{n-1}| - (1 + \alpha|m|)\big) \\ &\geq |y|\big(\alpha|y^{n-1}| - (1 + |m|)\big) \end{aligned}$$

$$|x_1| \geq |x|\left(\frac{\alpha|x^{n-1}|}{1+|m|} - 1\right)$$

Also, from assumption, $|x| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{n-1}}$ and hence $\left(\frac{\alpha|x^{n-1}|}{1+|m|} - 1\right) > 1$, thus there exists a real number $\gamma > 0$ such that $\left(\frac{\alpha|x^{n-1}|}{1+|m|} - 1\right) > 1 + \gamma$ and finally get $|x_1| > (1+\gamma)|x|$, continuing this process gives $|x_i| > (1+\gamma)^i|x|$.

Hence the orbit of x → ∞.

Also now $|x| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}, |r|\right\}$, with n ≥ 2 and $\alpha, \beta \in (0, 1]$ then $|x_i| \to \infty$ as $i \to \infty$.

Theorem 3.2

Suppose that $\{x_i\}$, i ∈ ℕ∪ {0} is a sequence in the Picard Ishikawa type orbit for the complex polynomial $P_{\mathbb{C}}(x) = x^n + m\,x + r$ where m, r ∈ ℂ, with n ≥ 2 such that $|x_i| \to \infty$ as $i \to \infty$, then $|x| \geq |r| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{n-1}}$ and $|x| \geq |r| > \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}$, $\alpha, \beta \in (0, 1]$.

Proof: Let $\{x_i\}$, i ∈ ℕ be a sequence in the Picard Ishikawa type orbit. Now firstly going to prove that $|x| \geq |r|$.
From given hypothesis, $|x_i| \to \infty$ as $i \to \infty$, the sequence $\{|x_i|\}$ is unbounded. Hence $|x_i| \geq |r|$ for all i ∈ ℕ ∪ {0} and therefore$|x| \geq |r|$. Let $P_{\mathbb{C}}(x) = x^n + m\,x + r$ where m, r ∈ ℂ, where x_0 = x, y_0 = y, z_0 = z, and t_0 = t then $|x| \geq |r|$implies that

$$|t| = |(1-\beta)x + \beta P_{\mathbb{C}}(x)| = |(1-\beta)x + \beta(x^n + m\,x + r)|$$
$$\geq |(1-\beta)x + \beta(x^n + m\,x)| - \beta|r|$$
$$\geq \beta|x^n| - (1 - \beta + \beta|m|)\,|x| - \beta|x|$$
$$\geq \beta|x^n| - (1+\beta|m|)\,|x|$$
$$|t| \geq |x|\big(\beta|x^{n-1}| - (1+\beta|m|)\big)$$
$$|t| = |x|(1+|m|)\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$$

implies that $|t| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$.

Now there are two possibilities either $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) \leq 1$ or $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > 1$.

If $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) \leq 1$ gives $\frac{\beta|x^{n-1}|}{1+|m|} \leq 2$ which implies that$|x^{n-1}| \leq \frac{2(1+|m|)}{\beta}$ and which gives $|x| \leq \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}$ a contradiction, $\{|x_i|\}$ is unbounded where i ∈ ℕ∪ {0}.Therefore $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > 1$.

Thus, $|x| > \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{n-1}}$ and $|t| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$implies that $|t| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > |x|$.
Also $\beta \leq 1$ and $|x| \geq |r|$, gives $|z| = |P_{\mathbb{C}}(t)| = |t^n + mt + r|$,

$$|z| \geq |t^n + mt| - |r|$$
$$\geq \beta|t^n| - |m||t| - |x|$$
$$\geq |t|\big(\beta|t^{n-1}| - (1+\beta|m|)\big)$$

As $\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > 1$, it gives $|t| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > |x|$. This results into $|z| = |x|(1+|m|)\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$

thus,
$$|z| = |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > |x|$$

proceeding in similar way, $|y| = |P_{\mathbb{C}}(z)| = |z^n + mz + r|$, and $|x| \geq |r|$ with $\beta \leq 1$ implies that $|y| \geq |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right)$ which leads to $|y| = |x|\left(\frac{\beta|x^{n-1}|}{1+|m|} - 1\right) > |x|$.
Lastly, $|x_1| = |(1-\alpha)y + \alpha P_{\mathbb{C}}(y)| = |(1-\alpha)y + \alpha(y^n + my + r)|$

$$\geq |(1-\alpha)y + \alpha(y^n + my)| - \alpha|r|$$
$$\geq \alpha|y^n| - (1 - \alpha + \alpha|m|)\,|y| - \alpha|y|$$
$$\geq \alpha|y^n| - (1+\alpha|m|)\,|y|$$
$$\geq |y|\big(\alpha|y^{n-1}| - (1+\alpha|m|)\big)$$
$$\geq |y|\big(\alpha|y^{n-1}| - (1+|m|)\big)$$
$$\geq |x|\big(\alpha|y^{n-1}| - (1+|m|)\big)$$
$$|x_1| \geq |x|\left(\frac{\alpha|x^{n-1}|}{1+|m|} - 1\right).$$

Proceeding in similar manner as above the one possibility that $\left(\frac{\alpha|x^{n-1}|}{1+|m|} - 1\right) > 1$and hence, $|x| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{n-1}}$.
This proves the result.

3.2 2nd Degree Complex Polynomials in a Picard Ishikawa Type Orbit with Escape Criterion

Now for 2nd degree complex polynomial $P_{\mathbb{C}}(x) = x^2 + m\,x + r$ where m, r ∈ ℂ, the following result gives escape criterion in a Picard Ishikawa type orbit.

Theorem 3.3

Suppose $|x| \geq |r| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right), \left(\frac{2(1+|m|)}{\beta}\right)\right\}$, $\alpha, \beta \in (0, 1]$. Define a sequence $\{x_i\}$, i ∈ ℕ as

$$\begin{cases} x_{i+1} = (1-\alpha)y_i + \alpha\, P_{\mathbb{C}} y_i, \\ y_i = P_{\mathbb{C}} z_i, \\ z_i = P_{\mathbb{C}} t_i, \\ t_i = (1-\beta)x_i + \beta\, P_{\mathbb{C}} x_i, \end{cases}$$

where x_0 = x, y_0 = y, z_0 = z, and t_0 = t. Then $|x_i| \to \infty$ as $i \to \infty$.

Proof: Let $P_{\mathbb{C}}(x) = x^2 + m\,x + r$, and from defined sequence $\{x_i\}$, i ∈ ℕ we have if $|x| \geq |r|$ then

$$|t| = |(1-\beta)x + \beta P_{\mathbb{C}} x| = |(1-\beta)x + \beta(x^2 + m\,x + r)|$$
$$\geq |(1-\beta)x + \beta(x^2 + m\,x)| - \beta|r|$$
$$\geq \beta|x^2| - (1 - \beta + \beta|m|)\,|x| - \beta|x|$$
$$\geq \beta|x^2| - (1+\beta|m|)\,|x|$$
$$|t| \geq |x|\big(\beta|x| - (1+\beta|m|)\big)$$

as $\beta \in (0, 1]$, implies that $-(1+\beta|m|) > -(1+|m|)$ which gives $|t| \geq |x|\big(\beta|x| - (1+|m|)\big)$
hence

$$|t| \geq |x|(1+|m|)\left(\frac{\beta|x|}{(1+|m|)} - 1\right)$$

Thus, $|t| \geq \frac{|t|}{(1+|m|)} \geq |x| \left(\frac{\beta|x|}{(1+|m|)} - 1\right)$ but $|x| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right), \left(\frac{2(1+|m|)}{\beta}\right)\right\}$ implies that $\left(\frac{\beta|x|}{(1+|m|)} - 1\right) > 1$.
Hence from above $|t| > |x|$.
Now from defined sequence $\{x_i\}$, i ∈ ℕ as z_0 = z with $\beta \in (0, 1]$, and $|x| \geq |r|$ also $|t| > |x|$ gives that

$$|z| = |P_{\mathbb{C}}(t)| = |t^2 + mt + r|,$$
$$|z| \geq |t^2 + mt| - |r|$$
$$\geq \beta|t^2| - |m||t| - |x|$$
$$\geq \beta|t^2| - |m||t| - |t|$$
$$\geq |t|(\beta|t| - (1 + \beta|m|))$$
$$\geq |t|(\beta|t| - (1 + |m|))$$

Hence $|z| \geq \frac{|z|}{(1+|m|)} \geq |t| \left(\frac{\beta|t|}{(1+|m|)} - 1\right)$ but $\left(\frac{\beta|x|}{(1+|m|)} - 1\right) > 1$ and $|t| > |x|$ implies that $\frac{\beta|t|}{(1+|m|)} > \frac{\beta|x|}{(1+|m|)}$ also $\left(\frac{\beta|t|}{(1+|m|)} - 1\right) > \left(\frac{\beta|x|}{(1+|m|)} - 1\right) > 1$. Therefore $|z| > |x|$.
Similarly, as y_0 = y with $\beta \in (0, 1]$ and $|x| \geq |r|$, also $|z| > |x|$ gives $|y| = |P_{\mathbb{C}}(z)| = |z^2 + mz + r|$,

$$|y| \geq |z^2 + mz| - |r|$$
$$\geq \beta|z^2| - |m||z| - |x|$$
$$\geq \beta|z^2| - |m||z| - |z|$$
$$\geq |z|(\beta|z| - (1 + \beta|m|))$$
$$\geq |z|(\beta|z| - (1 + |m|))$$

which gives $|y| \geq |z| \left(\frac{\beta|z|}{(1+|m|)} - 1\right)$. Then using $\left(\frac{\beta|x|}{(1+|m|)} - 1\right) > 1$ with $|z| > |x|$ gives that

$$|y| \geq |x| \left(\frac{\beta|x|}{(1+|m|)} - 1\right) > |x|$$

Lastly, as $\alpha \in (0, 1]$

$$|x_1| = |(1-\alpha)y + \alpha P_{\mathbb{C}}(y)| = |(1-\alpha)y + \alpha(y^2 + my + r)|$$
$$\geq |(1-\alpha)y + \alpha(y^2 + my)| - \alpha|r|$$
$$\geq \alpha|y^2| - (1 - \alpha + \alpha|m|)\,|y| - \alpha|y|$$
$$\geq \alpha|y^2| - (1 + \alpha|m|)\,|y|$$
$$\geq |y|(\alpha|y| - (1 + \alpha|m|))$$
$$\geq |y|(\alpha|y| - (1 + |m|))$$

$|x_1| \geq |y|(1 + |m|)\left(\frac{\alpha|y|}{(1+|m|)} - 1\right)$ but $|y| \geq |x|\left(\frac{\beta|x|}{(1+|m|)} - 1\right) > |x|$ implies that $|x_1| \geq |x|\left(\frac{\alpha|x|}{1+|m|} - 1\right)$.
From supposition that $|x| > \left(\frac{2(1+|m|)}{\alpha}\right)$ which implies $\left(\frac{\alpha|x|}{1+|m|} - 1\right)$. Thus there exists a real number $\gamma > 0$ such that $\left(\frac{\alpha|x|}{1+|m|} - 1\right) > 1 + \gamma$, finally get $|x_1| > (1 + \gamma)|x|$, continuing this process gives $|x_i| > (1 + \gamma)^i|x|$.
Hence the orbit of x → ∞.
Also now $|x| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right), \left(\frac{2(1+|m|)}{\beta}\right), |r|\right\}$, where $\alpha, \beta \in (0, 1]$ then $|x_i| \to \infty$ as $i \to \infty$.

3.3 3rd Degree Complex Polynomials in a Picard Ishikawa Type Orbit with Escape Criterion

Now for 3rd degree complex polynomial $P_{\mathbb{C}}(x) = x^3 + m\,x + r$ where m, r ∈ ℂ, the following result gives escape criterion in a Picard Ishikawa type orbit.

Theorem 3.4

Suppose $|x| \geq |r| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{2}}, \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{2}}\right\}$, $\alpha, \beta \in (0, 1]$. Define a sequence $\{x_i\}$, i ∈ ℕ as

$$\begin{cases} x_{i+1} = (1-\alpha)y_i + \alpha P_{\mathbb{C}} y_i, \\ y_i = P_{\mathbb{C}} z_i, \\ z_i = P_{\mathbb{C}} t_i, \\ t_i = (1-\beta)x_i + \beta P_{\mathbb{C}} x_i, \end{cases}$$

where x_0 = x, y_0 = y, z_0 = z, and t_0 = t. Then $|x_i| \to \infty$ as $i \to \infty$.

Proof: Let $P_{\mathbb{C}}(x) = x^3 + m\,x + r$, and from defined sequence $\{x_i\}$, i ∈ ℕ we have if $|x| \geq |r|$ then

$$|t| = |(1-\beta)x + \beta P_{\mathbb{C}}x| = |(1-\beta)x + \beta(x^3 + m\,x + r)|$$
$$\geq |(1-\beta)x + \beta(x^3 + m\,x)| - \beta|r|$$
$$\geq \beta|x^3| - (1 - \beta + \beta|m|)\,|x| - \beta|x|$$
$$\geq \beta|x^3| - (1 + \beta|m|)\,|x|$$
$$|t| \geq |x|(\beta|x^2| - (1 + \beta|m|))$$

as $\beta \in (0, 1]$,

$$|t| \geq |x|(\beta|x^2| - (1 + |m|))$$

Thus, $|t| \geq \frac{|t|}{(1+|m|)} \geq |x| \left(\frac{\beta|x^2|}{(1+|m|)} - 1\right)$ but $|x| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{2}}, \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{2}}\right\}$ implies that $\left(\frac{\beta|x^2|}{(1+|m|)} - 1\right) > 1$.
Hence from above $|t| > |x|$.
Now from defined sequence $\{x_i\}$, i ∈ ℕ as z_0 = z with $\beta \in (0, 1]$, and $|x| \geq |r|$ also $|t| > |x|$ gives that

$$|z| = |P_{\mathbb{C}}(t)| = |t^3 + mt + r|,$$

$$
\begin{aligned}
|z| &\geq |t^3 + mt| - |r| \\
&\geq \beta|t^3| - |m||t| - |x| \\
&\geq \beta|t^3| - |m||t| - |t| \\
&\geq |t|\big(\beta|t^2| - (1 + \beta|m|)\big) \\
&\geq |t|\big(\beta|t^2| - (1 + |m|)\big) \\
|z| &\geq |t|\left(\frac{\beta|t^2|}{(1+|m|)} - 1\right)
\end{aligned}
$$

Hence $|z| \geq \frac{|z|}{(1+|m|)} \geq |t|\left(\frac{\beta|t^2|}{(1+|m|)} - 1\right)$ but $\left(\frac{\beta|t^2|}{(1+|m|)} - 1\right) > 1$ and $|t| > |x|$ implies that $\frac{\beta|t^2|}{(1+|m|)} > \frac{\beta|x^2|}{(1+|m|)}$ also $\left(\frac{\beta|t^2|}{(1+|m|)} - 1\right) > \left(\frac{\beta|x^2|}{(1+|m|)} - 1\right) > 1$. Therefore $|z| > |x|$.

Similarly, as y_0 = y with $\beta \in (0, 1]$ and $|x| \geq |r|$, also $|z| > |x|$ gives $|y| = |P_{\mathbb{C}}(z)| = |z^3 + mz + r|$,

$$
\begin{aligned}
|y| &\geq |z^3 + mz| - |r| \\
&\geq \beta|z^3| - |m||z| - |x| \\
&\geq \beta|z^3| - |m||z| - |z| \\
&\geq |z|\big(\beta|z^2| - (1 + \beta|m|)\big) \\
&\geq |z|\big(\beta|z^2| - (1 + |m|)\big) \\
|y| &\geq |z|\left(\frac{\beta|z^2|}{(1+|m|)} - 1\right)
\end{aligned}
$$

which gives $|y| \geq |x|\left(\frac{\beta|x^2|}{(1+|m|)} - 1\right)$. Then using $\left(\frac{\beta|x^2|}{(1+|m|)} - 1\right) > 1$ with $|z| > |x|$ gives that

$$|y| \geq |x|\left(\frac{\beta|x^2|}{(1+|m|)} - 1\right) > |x|$$

Lastly, as $\alpha \in (0, 1]$

$$
\begin{aligned}
|x_1| = |(1-\alpha)y + \alpha P_{\mathbb{C}}(y)| &= |(1-\alpha)y + \alpha(y^3 + my + r)| \\
&\geq |(1-\alpha)y + \alpha(y^3 + my)| - \alpha|r| \\
&\geq \alpha|y^3| - (1 - \alpha + \alpha|m|)\,|y| - \alpha|y| \\
&\geq \alpha|y^2| - (1 + \alpha|m|)\,|y| \\
&\geq |y|\big(\alpha|y^2| - (1 + \alpha|m|)\big) \\
&\geq |y|\big(\alpha|y^2| - (1 + |m|)\big)
\end{aligned}
$$

$|x_1| \geq |y|(1 + |m|)\left(\frac{\alpha|y^2|}{(1+|m|)} - 1\right)$ but $|y| \geq |x|\left(\frac{\alpha|x^2|}{(1+|m|)} - 1\right) > |x|$ implies that $|x_1| \geq |x|\left(\frac{\alpha|x^2|}{(1+|m|)} - 1\right)$.

From supposition that $|x| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{2}}$ which implies $\left(\frac{\alpha|x^2|}{(1+|m|)} - 1\right)$. Thus there exists a real number $\gamma > 0$ such that $\left(\frac{\alpha|x^2|}{(1+|m|)} - 1\right) > 1 + \gamma$, finally get $|x_1| > (1 + \gamma)|x|$, continuing this process gives $|x_i| > (1 + \gamma)^i|x|$.

Hence the orbit of x → ∞.

Also now $|x| > \max\left\{\left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{2}}, \left(\frac{2(1+|m|)}{\beta}\right)^{\frac{1}{2}}, |r|\right\}$, where $\alpha, \beta \in (0, 1]$ then $|x_i| \to \infty$ as $i \to \infty$.

4. Graphical Construction of Mandelbrot Set and Julia Set.

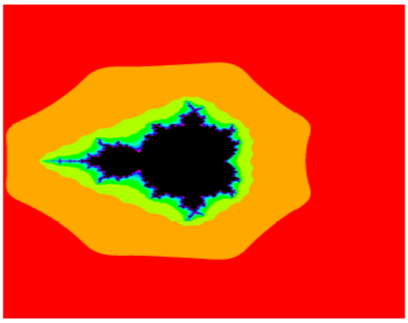
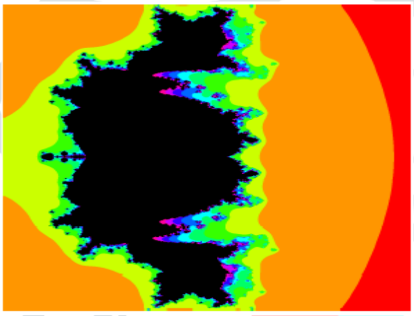
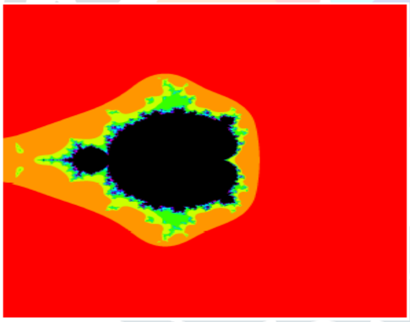
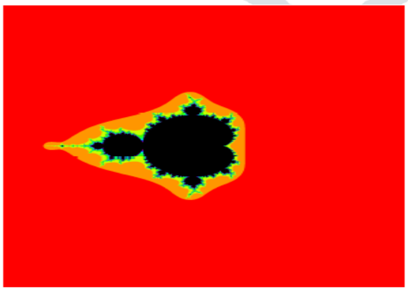
Now constructing the Mandelbrot set and Julia set graphically also in Picard Ishikawa type orbit for the complex polynomial, $P_{\mathbb{C}}(x) = x^n + mx + r$ with the help of MATLAB software's first introducing the corresponding algorithm, for this number of iterations are 10. Because of different attributes the color variations and change in shape occurs, here are presenting some of them and using this algorithms one can construct new fractals with variety in shape, patterns, colors, etc.

4.1Graphical Construction of Mandelbrot Set.

The following algorithm gives the construction of Mandelbrot sets.

```
for c ∈ A
   do
      r = Threshold radius
       i = 0
x_0 = critical point of P
       while i ≤ N
             z = P′(z)
             if |z| > r then
                break
             end
             i = i + 1
        end
    m = [(M-1) i/N]
    color c with colormap[m]
end
```

Here, P: ℂ → ℂ is complex polynomial, A - Area (A⊂ ℂ), $P'(z)$ – The iteration process, N – number of iterations, r , m ∈ ℂ where m is colormap [0…M - 1] – colormap with M colors.
The following are the area of Mandelbrot set represented graphically which is symmetric about origin for different variations of colors and other attributes for A = [-2, 2] × [-1.2, 2.5]

Sr. No.	α	β	Mandelbrot Set	Features
1	0.009	0.009		Color Variation
2	0.1	0.3		Stretching or Expansion
3	0.75	0.7		Compactness
4	1	1		Color Variation

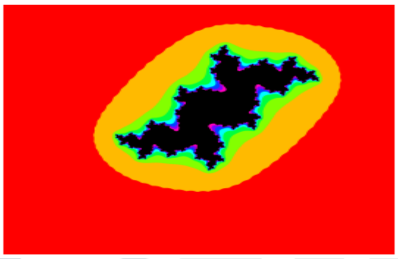
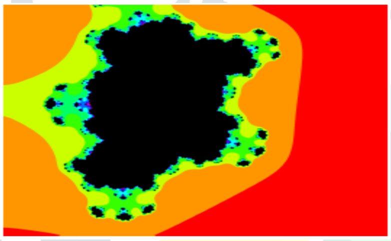
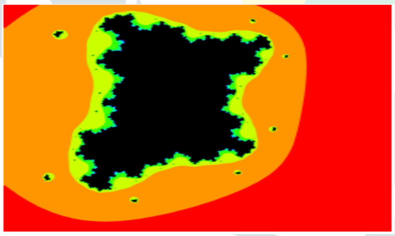
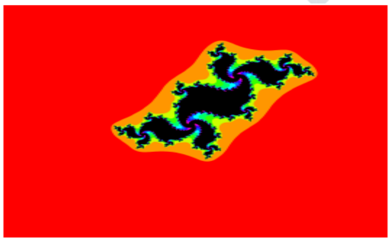
4.2 Graphical Construction of Julia Set.
The following algorithm gives the construction of Mandelbrot sets.

r = Threshold radius
for c ∈ A
do
i = 0
x_0 = critical point of P
while i ≤ N
z = P′(z)
if $|z|$ > r then

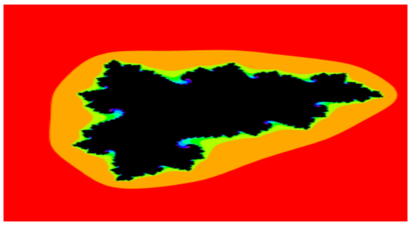
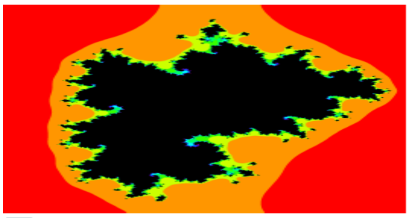
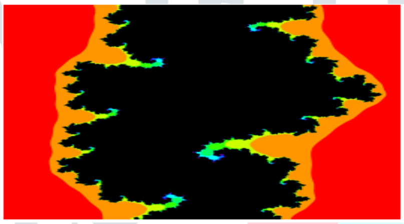
```
            break
         end
         i = i + 1
     end
   m = [(M-1)i/N]
   color c with colormap[m]
end
```

Here, P: ℂ → ℂ is complex polynomial, A - Area (A⊂ ℂ), $P'(z)$ – The iteration process, N – number of iterations, r, m ∈ ℂ, where m is colormap [0…M - 1] – colormap with M colors.

The following are the area of 2nd order Julia set represented graphically for different variations of colors and other attributes for A = [-2.5, 2.5] × [-2.1, 2.1] and the polynomial P(x) = x^2 + (- 0.5 + 0.7i) x + (- 0.01 + 0.18i) with one attracting fixed point p = - 0.1427 + 0.1019i

Sr. No.	α	β	2nd order Julia set	Features
1	10^{-10}	10^{-10}		Color Variation but same in shape and size as that of 4
2	0.11	0.18		Different shape and size because of more Color Variation as that of 3
3	0.2	0.097		Different shape and size because of more Color Variation as that of 2
4	1	1		Color Variation but same in shape and size as that of 1

The following are the area of 3rd order Julia set represented graphically for different variations of colors and other attributes for A = [-1.5, 1.5] × [-1.8, 1.8] and the polynomial P(x) = x^3 + (- 0.275+ 0.5i) x + (- 0.559 + 0.35i) with one attracting fixed point p = - 0.6434 + 0.2687i

Sr. No.	α	β	3rd order Julia Set	Features
1	10^{-10}	10^{-10}		Color Variation but same in shape and size as that of 4
2	0.08	0.09		More Color Variation as that of 2
3	0.1	0.2		Less Color Variation as that of 2
4	1	1		Color Variation but same in shape and size as that of 1

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