A Study on R-Near Ring

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Abstract : In this paper we introduce the concept of R-near ring. We define that a near ring N is a R-near ring if for every $a \in N$, there exists $x \in N$ such that $xax = xa$. The properties of R- near ring are discussed using the concept of zero divisors, ideal and subcommutativity. It is proved that every ideal and every left N-subgroup of any R-near ring without zero divisors is an R-near ring. It is also proved that any R-near ring is IFP near ring and subcommutative if it is Boolean; zero-symmetric if it is commutative also. A commutative zero symmetric Boolean near ring is always a R-near ring.

Keywords - N-subgroup, IFP, subcommutative, nil near ring, distributive, Boolean.

1.Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring.

Throughout this paper N stands for a right near ring $(N, +, .)$, with at least two elements and '0' denotes the identity element of the group $(N, +)$ and we write xy for $x.y$ for any two elements x, y of N. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that N is **zero symmetric**. An element a is said to be **nilpotent** if $a^k = 0$ for some positive integer k.

2.Preliminaries

Notation 2.1 [1]

E denotes the set of all idempotents of N ($a \in E$ iff $a^2 = a$)

Definition 2.2 [2]

If all nonzero elements of N are left (right) cancelable, we say that N fulfills the left (right) cancellation law.

Definition 2.3 [5]

An element $0 \neq x \in N$ is called a **right zero divisor** if there exists $0 \neq a \in N$ such that $ax = 0$.

Definition 2.4 [5]

An element $0 \neq x \in N$ is called a **left zero divisor** if there exists $0 \neq a \in N$ such that $xa = 0$.

Definition 2.5 [5]

A **zero divisor** is an element that is either a left (or) right zero divisor.

Definition 2.6 [2]

If $(N, .)$ is commutative we call N itself a **commutative near ring**.

Definition 2.7 [2]

A near ring N is **Boolean** if and only if for all $x \in N$: $x^2 = x$.

Definition 2.8

An additive group A of N is called a **left N-subgroup** if $NA \subseteq A$ where $NA = \{ra | a \in A, r \in N\}$.

Definition 2.9 [2]

Let N be a near ring and P a N-group. A normal subgroup I of $(N, +)$ is called an **ideal** of N if (i) $IN \subseteq I$ (ii) For all $n, n_1 \in N$ and for all $i \in I$, $n(n_1 + i) - nn_1 \in I$.

Definition 2.10 [2]

N is called a **nil near ring** if every element of N is nilpotent.

Definition 2.11 [4]

N is said to be **subcommutative** if $Na = aN$ for all $a \in N$.

Definition 2.12 [1]

N is called a **P_k near ring ($P_k{}'$ near ring)** if there exists a positive integer k such that $x^k N = xNx$ $(Nx^k = xNx)$ for all $x \in N$.

Definition 2.13 [2]

N is said to fulfill the **Insertion of Factors Property (IFP)** provided that for all a, b, n in N, $ab = 0 \Longrightarrow anb = 0$.

Definition 2.14 [2]

$N_d = \{n \in N | n(a + a') = na + na' \; for \; all \; a, a' \in N\}$ - the set of all distributive elements of N. N is called **distributive** if $N = N_d$.

Theorem 2.15 [2]

Let I be an ideal of N. N is nil if and only if I and N/I are nil.

Theorem 2.16 [2]

Let I be an ideal of N. Then N is zero symmetric if and only if I and N/I are zero symmetric.

3.R-near ring

Definition 3.1

N is called a **R-near ring** if for every $a \in N$ there exists $x \in N$ such that $xax = xa$.

Theorem 3.2

Let N be an R-near ring. (i) If $ax = 0$ then $xa = 0$. (ii) If $ax \in E$ then $xa \in E$. (iii) If the left cancellation law is valid in N, then $xa \in E$ implies $ax \in E$, for all $a \in N$ and for some $x \in N$.

Proof:

Let $a \in N$. Since N is an R-near ring, there exists $x \in N$ such that $xax = xa \longrightarrow (1)$. (i) If $ax = 0$ then from (1) we get, $xa = x0 = 0$. Thus $xa = 0$. (ii) If $ax \in E$ then $(ax)^2 = ax \longrightarrow (2)$. Now $(xa)^2 = (xa)(xa) = xa(xax) = x(ax)^2 = x(ax) = xax = xa$. That is $(xa)^2 = xa$ and hence $xa \in E$. (iii) If $xa \in E$, then $(xa)^2 = xa \longrightarrow (3)$. Now $x(ax)^2 = (xa)(xax) = (xa)(xa) = (xa)^2 = xa = xax$. That is $x(ax)^2 = xax$. Since the left cancellation law is valid in N, $(ax)^2 = ax$. Thus $ax \in E$.

Proposition 3.3

Let N be a R-near ring without nonzero zero divisors. If N is commutative then N is Boolean.

Proof:

Let $a \in N$. Since N is an R-near ring, there exists $x \in N$ such that $xax = xa$. Since N is commutative, $x(xa) = xa \Longrightarrow x^2a = xa \Longrightarrow (x^2 - x)a = 0$. Since N has no nonzero zero divisors, $x^2 - x = 0$. Consequently, N is Boolean.

Proposition 3.4

Let N be a R-near ring. If N has no zero divisors, then every left N-subgroup and every ideal of N is a R-near ring in its own right.

Proof:

Let M be an left N-subgroup of N and let $a \in M$. Since N is a R-near ring, there exists $x \in N$ such that $xax = xa$. Take $m = xa \in NM$. Since N is a left N-subgroup of N, $NM \subseteq M$. Therefore $m \in M$. Since N has no zero divisors, $m \neq 0$. Now $xmx = x(xa)x = x(xax) = x(xa) = xm$. Thus M is a R-near ring. Now, let I be an ideal of N and let $a \in I$. Since N is a R-near ring, there exists $x \in N$ such that $xax = xa$. Take $i = xa \in NI$. Since I is an ideal of N, $NI \subseteq I$. Therefore $i \in I$. Since N has no zero divisors, $m \neq 0$. Now $xix = x(xa)x = x(xax) = x(xa) = xi$. Thus I is a R-near ring.

Theorem 3.5

Let N be a nil near ring. Then N is a R-near ring if and only if N is zero symmetric.

Proof:

Take $a \in N$. Since N is a R-near ring, there exists $x \in N$ such that $xax = xa \longrightarrow (1)$. We shall prove that $xa^kx = xa^k \longrightarrow (2)$ for all positive integer k, we use induction on k. Equation (2) is true for $k = 1$. Assume that the result is true for $k = s - 1$. If $k = s$, then $xa^sx = (xa)(a^{s-1}x) = (xax)a^{s-1}x = xa(xa^{s-1}x) = xa(xa^{s-1}) = (xax)a^{s-1} = (xa)a^{s-1} = xa^s$. Thus $xa^kx = xa^k$ for all positive integers k. Since N is nil, $a^t = 0$ for some positive integer t. Since $xa^tx = xa^t$, $x0x = x0 \Longrightarrow x0 = 0$. Thus N is zero symmetric. Conversely, let $x \in N$. Since N is nil, there exists a positive integer such that $x^k = 0$. This implies $xa = 0$ where $a = x^{k-1}$. Therefore $xax = x(ax) = x(x^{k-1}x) = xx^k = x0 = 0 = xa$. Thus N is a R-near ring.

Theorem 3.6

Let N be a Boolean near ring. Each of the following statement implies that N is a R-near ring. (i) N is an IFP near ring with identity. (ii) $xN = xNx$ for all $x \in N$ (N is a P_1 near ring) (iii) N is subcommutative.

Proof:

(i) Let N be an IFP near ring with identity 1 and let $x \in N$. Since N is Boolean, $x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0$. Since N has IFP, $xa(x-1) = 0$ for all $a \in N$. This implies $xax - xa = 0 \Longrightarrow xax = xa$. (ii) Let $x \in N$. Since $xN = xNx$, for any $a \in N$, there exists $y \in N$ such that $xa = xyx$. Therefore $xax = (xa)x = (xyx)x = xyx^2 = xyx = xa$. Thus $xax = xa$. Hence N is a R-near ring. (iii) Let $x \in N$. Since N

is subcommutative, $xN = Nx$. Therefore for any $a \in N$, there exists $y \in N$ such that $xa = yx$. Therefore $xax = (xa)x = (yx)x = yx^2 = yx = xa$. That is $xax = xa$ for all $a \in N$. Thus N is a R-near ring.

Theorem 3.7

A commutative zero symmetric Boolean near ring is always a R-near ring.

Proof:

(i) Let N be a zero symmetric near ring. Let $x \in N$. If $x \neq 0$, we take $a = x$. Then $xax = x^2x = xx = xa$. That is $xax = xa$. If $x = 0$, then for any $a \in N$, $xax = 0$. Since N is commutative $xa = ax = 0$. Thus N is a R-near ring.

Theorem 3.8

Let N be a commutative near ring without nonzero zero divisors. Then N is a R-near ring if and only if N is Boolean.

Proof:

Let $a \in N$. For any $x \in N$, $xax = xa = (xa)x - xa = (ax)x - ax = ax^2 - ax = a(x^2 - x) = a0 = 0$. That is $xax - xa = 0 \Longrightarrow xax = xa$. Therefore N is a R-near ring. Conversely, Let N be Boolean. This implies $x^2 = x$ for all $x \in N \Longrightarrow x^2 - x = 0$. Since N has no zero divisors, $(x^2 - x)a = 0$ for all $a \in N$. $\Longrightarrow x^2a - xa = 0 \Longrightarrow x(xa) - xa = 0 \Longrightarrow x(ax) - xa = 0 \Longrightarrow xax = xa$. Thus N is a R-near ring.

Theorem 3.9

Let N be a nil near ring and I a nonzero ideal of N. Then N is a R-near ring if and only if I and N/I are R-near ring.

Proof:

Let I be a nonzero ideal of N. Since N is nil, by theorem 2.15 we get, I and N/I are nil. Let N be a R-near ring. Since N is nil, N is zero symmetric by theorem 3.5. Therefore by theorem 2.16, we get I and N/I are zero symmetric. By theorem 3.5, we get I and N/I are R-near rings. Conversely, let us assume that I and N/I are R-near rings. Therefore I and N/I are zero symmetric by theorem 3.5 and by theorem 2.16 N is zero symmetric. Again by theorem 3.5 N is a R-near ring.

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