# PROPERTIES OF GRAPHS IN GRAPH THEORY

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**ABSTRACT**

We have proposed studying properties that “almost always” hold. This phrase has meaning in the context of a probability model.

Definition. Given a sequence of probability spaces, let $q_n$ be the probability that property $Q$ holds in the nth space. Property $Q$ almost always holds if $\lim_{n\to\infty} q_n = 1$.

For us, the nth space is a probability distribution over $n$-vertex graphs. When property $Q$ almost always holds, we say almost every graph has property $Q$ Making aU graphs with vertex set $[n]$ equally likely is equivalent to letting each vertex pair appear as an edge with probability 1/2. Models where edges arise independently with the same probability are the most common for random graphs because they lead to the simplest computations. We allow this probability to depend on $n$.

Definition. Model A Given $n$ and $p = p(n)$ , generate graphs with vertex set $[n]$ by letting each pair be an edge with probability $p$, independently. Each graph with $m$ edges has probability $p^m(1-p)\binom{n}{2} -m$. The random variable $G^p$ denotes a graph drawn from this probability space. “*The* random graph” means Model A with $p = 1/2$, which makes all graphs with vertex set $[n]$ equally likely.

Computations are much simpler for graphs with a fixed vertex set ($''$labeled$''$ graphs) than for random isomorphism classes. Since inputs to algorithms are graphs with specified vertex sets, this model is consistent with applications.

We often measure running times of algorithms in terms of the number of vertices and number of edges; hence we may want to control the number of edges. This suggests a model in which the $n$-vertex labeled graphs with $m$ edges are equally likely. (We use $m$ to count edgesin this section because the number $e = 2.71828\ldots$ plays an important role in asymptotic arguments.)

Definition. Model B: Given $n$ and $m = m(n)$ , let each graph with vertex set $[n]$ and $m$ edges

occur with probability $\binom{N}{m}^{-1}$, where $N = \binom{n}{2}$. The random vańable $G^m$ denotes a graph generated in this way.

These two are the most common of many models studied. Model B seems more pertinent for applications. We ask questions hke as a function of $n$, how many edges are needed to make a graph almost surely connected?' In Model A we would say, "as a function of $n$, what edge probability is needed to make a graph almost surely connected?" Unfortunately, calculations needed to answer such questions are messier in Model B than in Model A.

Fortunately, Model B is accurately described by Model A when $n$ is large and $p = m/\binom{n}{2}$, because the actual number of edges generated in Model A is almost always very close to the resulting expectation $m$. The correspondence is valid for most properties of interest. The proof of this requires detailed use of the binomial distribution for the number of edges. A graph property $Q$ is convex if $G$ satisfies $Q$ whenever $F \subseteq G \subseteq H$ and $F, H$ satisfy $Q$.

**Key words:** partition , probability , standard deviation , threshold , composition, binomial distribution.

## INTRODUCTION

Theorem. (Bollobás [1985, p34-35]) If $Q$ is convex and $p(1-p)\binom{n}{2} \to \infty$, then almost every $G^p$ satisfies $Q$ if and only if, for every fixed $x$, almost every $G^m$ satisfies $Q$, where $m = \lfloor p\binom{n}{2} + x\left[p(1-p)\binom{n}{2}\right]^{1/2}\rfloor$.

Theorem. (Gilbert ) When $p$ is constait, almost every $G^p$ is connected.

Proof. We can make $G$ disconnected by picking a vertex partition into two sets and forbidding edges between the two sets. Occurrence of edges within the sets is irrelevant. We bound the Rrobability $q_n$ that $G^p$ is disconnected by summing $P([S,\overline{S}] = \emptyset)$ over aU bipartitions $S, \overline{S}$. Graphs with many components are counted many times. When $|S1 = k$, there are $k(n-k)$ possible edges in $[S,\overline{S}]$. Each has probability $1-p$ of not appearing, independently, so $P([S,\overline{S}] = \emptyset) = (1-p)^{k(n-k)}$. Considering all $S$ generates each partition from each side, so

$q_n \leq \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(1-p)^{k(n-k)}$ .

This formula is symmetric in $k$ and $n-k$; hence $q_n$ is bounded by $\sum_{k=1}^{\lfloor n/2 \rfloor}\binom{n}{k}(1-p)^{k(n-k)}$. We loosen the bound to simpli6$r$ it. Using $\binom{n}{k} < n^k$ and $(1-p)^{n-k} \leq (1-p)^{n/2}$ (for $k \leq n/2$) yields $q_n < \sum_{k=1}^{\lfloor n/2 \rfloor}\left(n(1-p)^{n/2}\right)^k$. For large enough $n$, we have $n(1-p)^{n/2} < 1$. This makes our bound the initial portion of a convergent geometric series. We obtain $q_n < x/(1-x)$ , where $x$a $= n(1-p)^{n/2}$. Since $n(1-p)^{n/2} \to 0$ when $p$ is constant, our bound on $q_{n,}$. approaches 0 as $n \to \infty$.

We avoid struggling with probability formulas by introducing integer valued random variables and techniques involving expectation. If $X$ is a nonnegative random variable such that $X = 0$ when $G^p$ has property $Q$, then $E(X) \to 0$ implies that almog every $G^p$ satisfies $Q$. This is a special case of the following lemma. We prove it only for integer variables, but it also holds for continuous variables.

Lemma. (Markov's Inequality) If $X$ takes only nonnegative values, then $P(X \geq t) \leq E(X)/t$. In particular, if $X$ is integer-valued, then $E(X) \to 0$
implies $P(X = 0) \to 1$.
Proof. $E(X) = \sum_{k\geq 0} k\, p_k \geq \sum_{k\geq t} k\, p_k \geq t\sum_{k\geq t} p_k = tP(X \geq t\rangle$.

For connectedness, we can define $X(G^p)$ by $X---1$ if $G$ is disconnected and $X = 0$ otherwise. The expectation ofan indicator vanable is the probability that it equals 1. We proved $P(X = 1) \to 0$ (when $p$ is constant) to prove that almost every $G^p$ is connected. With a different random variable we can simplify the proof and strengthen the result. We still want $G$ to satisfy $Q$ if $X = 0$ (in order to apply Markov's Inequality), but we don't need $(X = 0) \Leftrightarrow$($G$ satisfies $Q$). We use a sum $X$ ofmany indicator variables, such that $G$ satisfies $Q$ if $X = 0$. The linearity of expectation and convenience of $E(X_i) = P(X_i = 1\rangle$ for the indicator variables simplify the task of proving $E(X) \to 0$.

Theorem. If $p$ is constant, then almost every $G_p$ has diameter 2 (and hence is connected).

Proof Let $X(G^p)$ be the number of unordered vertex pairs with no common neighbor. If there are none, then $G_t$, is connected and has diameter 2. By Markov's Inequality, we need only show $E(X) \to 0$. We express $X$ as the sum of $\binom{n}{2}$ indicator variables $X_{i.j}$, one for each vertex pair $\{v_i, v_j\}$, where $X_{i.j} = 1$ if and only if $v_j$, $v_j$ have no common neighbor.

When $X_{i\backslash j} = 1$, the $n-2$ other vertices fail to have edges to both ofthese, so

$P(X_{i.j} = 1) = (1-p^2)^{n-2}$ and $E(X) = \binom{n}{2}(1-p^2)^{n-2}$. When $p$ is fixed, $E(X) \to 0$, and hence almost every $G_p$ has diameter 2.

The intuition behind this argument, made precise by Markov's Inequality, is that if we expect almost no bad pairs, then almost every graph has none. The summation disappears, and for the limit we need only know that $(1-p^2)^{n-2}$ tends to 0 faster than any polynomial function of $n$.

THRESHOLD FUNCTIONS

Roughly speaking, random graphs with constant edge probability are connected because they have many more edges than needed to be connected. To improve above Theorem , we want to make $p(n)$ as small as possible to have almost every $G^p$ connected. We need the notion of a threshold probability function. By the relationship between Model A and Model B, a threshold edge probability also yields a threshold number of edges.

Definition. A monotone property is a graph property preserved by addìtion of edges. A threshold probability function for a monotone property $Q$ is a function $t(n)$ such that $p(n)/t(n) \to 0$ implies that almost no $G^p$ satisfies $Q$, and $p(n)/t(n) \to \infty$ implies that almost every $G^p$ satisfies $Q$. Threshold edge fimction is defined similarly for Model B.

Definition. The $r$th moment of $X$ is the expectation of $X^r$. The variance of $X$, wntten $Var(X)$ , is the quantity $E\left[\left(X - E(X)\right)^2\right]$. The standard deviation of $X$ is the square root of $Var(X)$ .

Lemma. ( $Second\ Moment\ Method$) If $X$ is a random variable

$P(X=0) \leq \frac{E(X^2)-E(X)^2}{E(X)^2}$. Inparticular, $P(X=0) \to 0$ when $\frac{E(X^2)}{E(X)^2} \to 1$ .

Proof. Applied to the variable $(X - E(X))^2$ and the value $t^2$, Markov's Inequality yields $P\left[(X-E(X))^2 \geq t^2\right] \leq E\left[(X-E(X))^2\right]/t^2$.We rewńte this as $P[|X-E(X)1 \geq t] \leq Var(X)/t^2$ (Chebyshev's Inequality). Since

$$E\left[(X-E(X))^2\right] = E\left[X^2 - 2XE(X) + (E(X))^2\right] = E(X^2) - (E(X))^2,$$

Chebyshev's Inequality becomes $\dot{P}[|X-E(X)| \geq t] \leq (E(X^2)-E(X)^2)/t^2$. Since $X=0$ only when $|X-E(X)1 \geq E(X)$ , setting $t=E(X)$ completes the proof ·

Intuitively, if the mean grows and the standard deviation grows more slowly, then all the probability is pulled away from 0, and $P(X=0) \to 0$ results. We illustrate the method by considenng the disappearance of isolated vertices. Since a connected graph has no isolated vertices, a threshold for connectedness must be at least as large as a threshold for disappearance ofisolated vertices. The computatiōns for the latter are simpler, because we can express this condition using a sum of identically distributed indicator variables with easily computed expectations. In fact, both properties have the same threshold, since it happens that at the threshold almost every graph consists of one huge component plus isolated vertices.

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