Fuzzy Metric spaces and related fixed point theorems

S. M. Yunus,
Department of Mathematics, Aligarh Muslim University, UP, India.

Abstract: In this present work I'll try to establish several fixed point theorems for a new class of self-maps in M -complete fuzzy metric spaces and compact fuzzy metric spaces, respectively.

INTRODUCTION

In 1975, Kramosil and Michálek [1] first introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. Clearly, this work provides an important basis for the construction of fixed point theory in fuzzy metric spaces. Afterwards, Grabiec [2] defined the completeness of the fuzzy metric space (now known as a G-complete fuzzy metric space; see Ref. [3]) and extended the Banach contraction theorem to G-complete fuzzy metric spaces. Following Grabiec's work, Fang [4] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra et al. [5] also obtained several common fixed point theorems for asymptotically commuting maps in the same space, which generalize several fixed point theorems in metric, Menger, fuzzy and uniform spaces. Besides these works based on the G-complete fuzzy metric space, George and Veeramani [6] modified the definition of the Cauchy sequence introduced by Grabiec [2] because even R is not complete with Grabiec's completeness definition. Meanwhile, they slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek [1] and then defined a Hausdorff and first countable topology. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani (now known as an M -complete fuzzy metric space; see Ref. [7]) has emerged as another characterization of completeness, and some fixed point theorems have also been constructed on the basis of this metric space. From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces [3,8–17]. Note that every G-complete fuzzy metric space is M -complete; the construction of fixed point theorems in M -complete fuzzy metric spaces seems to be more valuable.

The purpose of this work is to propose a new class of self-maps by using a ϕ-function. More importantly, we prove the existence of a fixed point for these self-maps in M -complete fuzzy metric spaces and compact fuzzy metric spaces in the senses of George and Veeramani, respectively.

LITERATURE REVIEW

Now, we begin with some basic concepts. Let N denote the set of all positive integers.

Definition 2.1 (Schweizer and Sklar [18]). A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t -norm if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative; (TN-2) $*$ is continuous;

(TN-3) $a * 1 = a$ for every $a \in [0, 1]$;

(TN-4) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2 (George and Veeramani [6]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(FM-1) $M(x, y, t) > 0$;

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$; (FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM-5) $M(x, y, \bullet): (0, \infty) \to (0, 1]$ is continuous.

Definition 2.3 (George and Veeramani [6], Gregori and Sapena [3]). Let (X, M , ∗) be a fuzzy metric space. Then:

(i) A sequence {xn } is said to converge to x in X , denoted by xn → x, if and only if limn→∞ M (xn , x, t) = 1 for all t > 0, i.e. for each r ∈ (0, 1) and t > 0, there exists n0 ∈ N such that M (xn , x, t) > 1 − r for all n ≥ n0 .
(ii) A sequence {xn } in X is an M -Cauchy sequence if and only if for each ϵ ∈ (0, 1), t > 0, there exists n0 ∈ N such that
M (xm , xn , t) > 1 − ϵ for any m, n ≥ n0 .
(iii) The fuzzy metric space (X, M, ∗) is called M -complete if every M -Cauchy sequence is convergent.

Definition 2.4 A fuzzy metric space (X, M, ∗) is compact if every sequence in M has a convergent subsequence.

THE MAIN RESULTS

In this section, I will establish several fixed point theorems for self-maps of the M -complete fuzzy metric space and compact fuzzy metric space. In these metric spaces, a function ϕ: [0, 1] → [0, 1] which is used by altering the distance between two points satisfies the following properties:

(P1) ϕ is strictly decreasing and left continuous; (P2) ϕ(λ) = 0 if and only if λ = 1.
Obviously, we obtain that limλ→1− ϕ(λ) = ϕ(1) = 0.

Theorem 3.1. Let (X , M , *) be an M -complete fuzzy metric space and T a self-map of X and suppose that ϕ : [0, 1] → [0, 1] satisfies the foregoing properties (P1) and (P2). Furthermore, let k be a function from (0, ∞) into (0, 1). If for any t > 0, T satisfies the following condition:

$$\phi(M(Tx, Ty, t)) \leq k(t) \cdot \phi(M(x, y, t)), \qquad (1)$$

Where x, y * X and x = y, then T has a unique fixed point.

Proof. Let x_0 be a point in X. Define xn+1 = Txn and τn (t) = M (xn , xn+1 , t) for all n * N ∪ {0}, t > 0. Now we first prove that T has a fixed point. The proof is divided into two cases.
Case1. If there exists n_0 * N ∪ {0} such that xn_0 +1 = x n_0, i.e., Tx n_0 = x n_0, then it follows that x n_0 is a fixed point of T .
Case 2. We assume that 0 < τn (t) < 1 for each n. That is to say, the relationship xn = xn+1 holds true for each n. From (1), for every t > 0, we can obtain

$$\phi(\tau_n(t)) = \phi(M(x_n, x_{n+1}, t)) = \phi(M(Tx_{n-1}, Tx_n, t)) \leq k(t) \cdot \phi(\tau_{n-1}(t)) < \phi(\tau_{n-1}(t)). \qquad (2)$$

Since ϕ is strictly decreasing, it is easy to show that {τn (t)} is an increasing sequence for every t > 0 with respect to n. We put limn→∞ τn (t) = τ (t) and suppose that 0 < τ (t) < 1. By (2), then τn (t) ≤ τ (t) implies that

$$\phi(\tau_{n+1}(t)) \leq k(t) \cdot \phi(\tau_n(t)). \qquad (3)$$

For every t , by supposing that n → ∞, since ϕ is left continuous, we have

$$\phi(\tau(t)) \leq k(t) \cdot \phi(\tau(t)) < \phi(\tau(t)), \qquad (4)$$

Which is a contradiction.
Hence τ (t) ≡ 1. That is, the sequence {τn (t)} converges to 1 for any t > 0.

Next, we show that the sequence $\{xn\}$ is an M -Cauchy sequence. Suppose that it is not. Then there exist $0 < \epsilon < 1$ and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that for every $n \in N \cup \{0\}$ and $t > 0$, we obtain that

$p(n) > q(n) \geq n$, $M(xp(n), xq(n), t) \leq 1 - \epsilon$,

$M(xp(n)-1, xq(n)-1, t) > 1 - \epsilon$, and $M(xp(n)-1, xq(n), t) > 1 - \epsilon$. (5)

For each $n \in N \cup \{0\}$, we suppose that $sn(t) = M(xp(n), xq(n), t)$; then we have

$$1 - \epsilon \geq sn(t) = M(xp(n), xq(n), t) \geq M(xp(n)-1, xp(n), t/2) * M(xp(n)-1, xq(n), t/2) > \tau p(n)(t/2) * (1 - \epsilon). \quad (6)$$

Since $\tau n(t/2) \to 1$ as $n \to \infty$ for every t, supposing that $n \to \infty$, we note that $\{sn(t)\}$ converges to $1 - \epsilon$ for any $t > 0$.

Moreover, by (1), we have

$$\phi(M(xp(n), xq(n), t)) \leq k(t) \bullet \phi(M(xp(n)-1, xq(n)-1, t)) < \phi(M(xp(n)-1, xq(n)-1, t). \quad (7)$$

According to the monotonicity of ϕ, we know that

$M(xp(n), xq(n), t) > M(xp(n)-1, xq(n)-1, t)$ for each n. Thus, on the basis of the formula (5) we can obtain

$1 - \epsilon \geq M(xp(n), xq(n), t) > M(xp(n)-1, xq(n)-1, t) > 1 - \epsilon$. (8)

Clearly, this leads to a contradiction.

In particular, we consider another case. That is, there exists $n_0 \in N \cup \{0\}$ such that

$M(xm, xn, t) \leq 1 - \epsilon$ for all $m, n \geq n_0$.

Obviously, for any $p \in N$, we know that $M(x\, n_0 + p+2, x\, n_0 + p+1, t) \leq 1 - \epsilon$.

As ϕ is monotonic, it is easy to see that the sequence $\{M(xn0 + p+2, xn0 + p+1, t)\}$ is a monotone and bounded sequence with respect to p. Therefore, there exists $\alpha \in (0, 1 - \epsilon]$ such that

$\lim_{p\to\infty} M(xn0 + p+2, xn0 + p+1, t) = \alpha$ for all $t > 0$. Thus, we can obtain

$$\phi(M(xn0 + p+2, xn0 + p+1, t)) \leq k(t) \bullet \phi(M(xn0 + p+1, xn0 + p, t)). \quad (9)$$

By supposing that $p \to \infty$, we have $\phi(\alpha) \leq 0$, which is also a contradiction.

Hence $\{xn\}$ is an M -Cauchy sequence in the M -complete fuzzy metric space X .

Therefore, we conclude that there exists a point $x \in X$ such that $\lim_{n\to\infty} xn = x$.

Now we will show that x is a fixed point of T . Since $0 < \tau n(t) < 1$, there exists a subsequence $\{xr(n)\}$ of $\{xn\}$ such that $xr(n) = x$ for every $n \in N$. From (2), we can obtain

$0 \leq \phi(M(xr(n)+1, Tx, t)) = \phi(M(Txr(n), Tx, t)) \leq k(t) \bullet \phi(M(xr(n), x, t))$. (10) By supposing that $n \to \infty$ in (10), we have

$$0 \leq \phi(M(x, Tx, t)) \leq k(t) \bullet \phi(M(x, x, t)) = k(t) \bullet \phi(1) = 0. \quad (11)$$

So we can get $\phi(M(x, Tx, t)) = 0$.

According to the property (P2), it is easy to show that $M(x, Tx, t) = 1$, i.e., $Tx = x$. Furthermore, we claim that x is the unique fixed point of T.

Assume that y (=x) is another fixed point of T, we then obtain

$$\phi(M(x, y, t)) = \phi(M(Tx, Ty, t)) \leq k(t) \bullet \phi(M(x, y, t)) < \phi(M(x, y, t)), \quad (12)$$

Which is a contradiction. The proof of the theorem is now completed.

Example 1. Let X be the subset of R2 defined by

X = {A, B, C, D, E },
Where A = (0, 0), B = (1, 0), C = (1, 2), D = (0, 1), E = (1, 3). $\phi(\tau) = 1 - \sqrt{\tau}$ for all $\tau \in [0, 1]$ and $M(x, y, t) = e^{-2d(x,y)/t}$ for all $t > 0$,
where d(x, y) denotes the Euclidean distance of R2 . Clearly, $(X, M, *)$ is an M -complete fuzzy metric space with respect to the t -norm: $a * b = ab$.

Let $T : X \to X$ be given by
T (A) = T (B) = T (C) = T (D) = A, T (E) = B.

Define function $k : (0, \infty) \to (0, 1)$ as

$$k(t) = \begin{cases} 1 - e^{-4/t} & 0 < t \leq 2 \\ \dfrac{t}{t+1} & t > 2. \end{cases}$$

One can see that the function ϕ satisfies (P1) and (P2), and the function k also satisfies the formula (1). Now, all the hypotheses of Theorem 3.1 are satisfied and thus T has a unique fixed point, that is x = A.

Corollary 3.2. Let $(X, M, *)$ be a compact fuzzy metric space and T a continuous self-map of X and suppose that $\phi : [0, 1] \to [0, 1]$ satisfies the foregoing (P1) and (P2). Furthermore, let k be a function from $(0, \infty)$ into $(0, 1)$. If for any $t > 0$, T satisfies the following condition:

$$\phi(M(Tx, Ty, t)) \leq k(t) \cdot \phi(M(x, y, t)),$$

Where $x, y \in X$ and $x = y$, then T has unique fixed point.

Proof. This theorem can be proved by the same method as was employed in Theorem 3.

RESULTS AND DISCUSSION

In this work, we proposed a new class of self-maps by altering the distance between two points in fuzzy environment, in which the ϕ-function was used. On the basis of this kind of self-map, we further proved some fixed point theorems in M - complete fuzzy metric spaces and compact fuzzy

metric spaces. Obviously, the present investigation enriches our knowledge of fixed points in M -fuzzy metric spaces.

REFERENCES

[1] I. Kramosil, J. Michálek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 336–344.

[2] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385–389.

[3] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002) 245–252.

[4] J.X. Fang, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 46 (1992) 107–113.

[5] S.N. Mishra, N. Sharma, S.L. Singh, Common fixed points of maps on fuzzy metric spaces, International Journal of Mathematics and Mathematical Sciences 17 (1994) 253–258.

[6] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[7] R. Vasuki, P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets and Systems 135 (2003) 415–417.

[8] M. Imdad, J. Ali, Some common fixed point theorems in fuzzy metric spaces, Mathematical Communications 11 (2006) 153–163.

[9] S. Kumar, D. Miheţ, G-completeness and M -completeness in fuzzy metric spaces: a note on a common fixed point theorem, Acta Mathematica Hungarica 126 (2010) 253–257.

[10] D. Miheţ, A Banach contraction theorem in fuzzy metric spaces, Fuzzy Sets and Systems 144 (2004) 431–439.

[11] D. Miheţ, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 915–921.

[12] D. Miheţ, A class of contractions in fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010) 1131–1137.

[13] S. Sedghi, I. Altun, N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Analysis 72 (2010) 1298–1304.

[14] S. Sharma, Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 127 (2002) 345–352.

[15] B. Singh, M.S. Chauhan, Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 471–475.

[16] R. Vasuki, A common fixed point theorem in a fuzzy metric space, Fuzzy Sets and Systems 97 (1998) 395–397.

[17] G. Yun, S. Hwang, J. Chang, Fuzzy Lipschitz maps and fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010) 1117–1130.

[18] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics 10 (1960) 385–389.