Some Results on Prime Graphs

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Key Words:
Prime Graphs, Prime Labelling, Strongly Prime Graphs, Umbrella graphs, Flower pot graphs

Abstract:
A graph $G = (V, E)$ with n vertices is said to admit a Prime Labeling if its vertices can be labelled with distinct positive integers not exceeding n such that the label of each pair of adjacent vertices are relatively prime. A graph G is said to be a strongly prime graph if for any vertex v of G there exist a prime labeling f with $f(v) = 1$. In this paper, we prove that the Umbrella graph $U(m, n)$ is strongly prime graph for $m = p$ and $n = p + 1$ and that the Flower Pot graph is a strongly prime graph.

1. INTRODUCTION:

Unless otherwise specified, we consider only simple, connected and non-trivial graphs. Two integers a and b are said to be relatively prime if their greatest common divisor is 1. The concept of prime labeling was introduced by Roger Entringer and was discussed Tout [4]. Many researchers have studied prime graphs. Fu and Huany [2] proved that the path P_n on n vertices is a prime graph. Deresky [1] proved that the cycle C_n on n vertices is a prime graph. Lee [3] proved that Wheel graph, W_n is aprime graph iff n is even. Vidya and Udayan [5] introduced a related concept, called strongly prime graph. They proved the graphs C_n, P_n, and $K_{1,n}$ are strongly prime graphs and W_n is strongly prime graph for every even integer $n \geq 4$.

Meena and Amuda [6] proved that some classes of graphs such as the flower pot graph, Umbrella graph, coconut tree graph, shell graph, corona of a shell graph, corona of a wheel graph, corona of a gear graph, butterfly graph are prime graphs. Meena and Kavitha [7, 8] proved that some special classes of graphs such as gearwheel graphs, helm graphs, corona of ladder graphs, corona of triangular snake, corona of quadrilateral snake are strongly prime graphs. These works are very much significant as they try to connect Graph Theory and Number Theory as pointed by Suresh Kumar [9]. In this paper, we will prove that for any prime number, p, the Umbrella graph $U(m, n)$ with $m = p$ and $n = p + 1$ is a strongly prime graph. We will also prove that the Flower Pot graph is a strongly prime graph.

Now, we recall the important concepts in Prime Graph Theory. The Greatest Common Divisor (GCD) of two integers, a, b is denoted by(a, b).

Definition.1.1. Let G be a graph with p vertices. A bijection $f : V(G) \to \{1,2,3 \dots p\}$ is called a prime labeling if for each edge $e = uv$, $\big(f(u), f(v)\big) = 1$. A graph which admits prime labeling is called a prime graph.

Definition 1.2. A graph G is said to be a strongly prime graph if for any vertex v of G there exist a prime labeling f satisfying $f(v) = 1$.

Definition 1.3. Fan graph f_m is defined as the graph $P_m + K_1$ where P_m is the path on m vertices and K_1 is the empty graph on one vertex.

Definition .1.4. Umbrella graph $U(m, n)$ is the graph obtained by joining a path P_n with the central vertex of a fan f_m.

Definition 1.5. Two prime numbers p and q are called adjacent primes if $q = p + 2$. Two prime numbers p and q are called pseudo primes if $q - p > 2$.

Result 1.6 (Bertrand's Postulate) For every integer $n > 1$, there is always exist at least one prime number p such that $n < p < 2n$.

2. MAIN RESULTS:

We are proving that the umbrella graph $U(m, n)$ is a strongly prime graph for $m = p$ and $n = p + 1$. We need the following result.

Lemma 2.1 For any prime number p, $(p + 1, 2p + 1) = 1$.

Proof:

Let $(p + 1, 2p + 1) = d$. Then d devides $p + 1$ and $2p + 1$.

Thus, $p + 1 = dk_1$, $2p + 1 = dk_2$, for some integer k_1 and k_2

Hence, $1 = d2k_1 - dk_2 = d(2k_1 - k_2)$, Since $(2k_1 - k_2)$ is an integer, we have $d = 1$.

Theorem 1: The umbrella graph $U(m,n)$ is a strongly prime graph for $m = p$ and $n = p + 1$.

Proof: Let $G = U(m,n)$ be the umbrella graph with $m + n$ vertices, $v_1, v_2, \dots\dots v_m, u_1, u_2, \dots\dots u_n$. Let $v_1, v_2, \dots\dots v_m$ be the vertices of the fan f_m and $u_1, u_2, \dots\dots u_n$ be the vertices of the path P_n, where u_1 is the central vertex. Thus, the vertex set, $V(G) = \{v_1, v_2, \dots\dots v_m, u_1, u_2, \dots\dots u_n\}$ and the edge set $E(G) = \{v_i v_{i+1} \mid 1 \le i \le m - 1\} \cup \{u_i u_{i+1} \mid 1 \le i \le n - 1\} \cup \{u_1 v_i \mid 1 \le i \le m\}$.

Let v be the vertex for which we assign label 1 in our labeling. Then we have two cased to the following cases.

Case.1: p is the lowest prime number in each pair of adjacent primes.

We have three subcases to consider.

Subcase.1: Let $v = u_1$

Define a labeling $f: V(G) \to \{1,2,3, \dots\dots 2p + 1\}$ by

$$f(u_1) = 1$$
$$f(u_i) = i + m \; for \; 2 \le i \le n$$
$$f(v_i) = i + 1 \; for \; 1 \le i \le m$$

Then for any edge, $v_i v_{i+1}$, $\gcd(f(v_i), f(v_{i+1})) = 1$. Since they are consecutive integers.

Similarly for any edge, $u_i u_{i+1}$, $\gcd(f(u_i), f(u_{i+1}) = 1$.

Also for the edge $u_1 v_i$, $\gcd(f(u_1), f(v_i)) = \gcd(1, f(v_i))$ =1. Thus it is a prime labelling.

Subcase.2: If $v = u_j$ for some $j = 2,3 \dots . n$.

Then define the labeling $f: V(G) \to \{1,2,3, \dots\dots 2p + 1\}$ by

$$f(u_i) = \begin{cases} p + 2 & if \; i = 1 \\ p + i + 1 & if \; 1 < i < j \\ 1 & if \; i = j \\ p + i & if \; j + 1 \le i \le p + 1 \end{cases}$$

$$f(v_i) = i + 1 \qquad for \; 1 \le i \le p$$

For edges $\{u_i u_{i+1}\}, for \; 1 \le i \le p$, except $u_{j-1} u_j \; and \; u_j u_{j+1}$, $gcd(f(u_i), f(u_{i+1})) = 1$, since they are consecutive integers.

For the edge $u_{j-1} u_j$, $\gcd\left(f(u_{j-1}), f(u_j)\right) = \gcd(f(u_{j-1}), 1) = 1$. Similarly for the edge $u_j u_{j+1}$,

For the edges $\{v_i v_{i+1}\} for \; 1 \le i \le p$, $\gcd(f(v_i), f(v_{i+1})) = (i + 1, i + 2) = 1$.

For the edges $\{u_1 v_i\}, for \; 1 \le i \le p$,$\gcd(f(u_1), f(v_i)) = \gcd(p + 2, i + 1) = 1$, since $p + 2$ is prime and maximum value of $i + 1 \; is \; p + 1$ .Thus f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j \; where \; j = 1,2,3 \dots . n$

Subcase.3: If $v = v_j$ for some $j = 1,2,3 \dots\dots m$

Then define the labeling $f: V(G) \to \{1,2,3, \dots\dots 2p + 1\}$ by

$$f(v_i) = p - j + i + 1 \; for \; 1 \le i \le j - 1$$
$$f(v_{j+i}) = i + 1, for \; 0 \le i \le p - j$$

For edges $\{v_i v_{i+1}\}, for \; 1 \le i \le p - 1$, except $v_{j-1} v_j \; and \; v_j v_{j+1}$, $gcd(f(v_i), f(v_{i+1})) = 1$, since they are consecutive integers.

For the edge $v_{j-1} v_j$, $\gcd\left(f(v_{j-1}), f(v_j)\right) = \gcd(f(v_{j-1}), 1) = 1$. Similarly, for the edge $v_j v_{j+1}$,

$$f(u_1) = p + 2$$
$$f(u_2) = p + 1$$
$$f(u_3) = p + p + 1 = 2p + 1$$
$$f(u_4) = p + p = 2p$$
$$f(u_5) = p + p - 1$$
$$\vdots$$
$$f(u_p) = p + 4$$
$$f(u_{p+1}) = p + 3$$

For the edges $\{u_i u_{i+1}\} \; for \; 1 \le i \le p$, except $u_2 u_3$ $\gcd(f(u_i), f(u_{i+1})) = 1$, since they are consecutive integers. For the edge $u_2 u_3$, by lemma 1, $\gcd(f(u_2) f(u_3)) = \gcd(p + 1, 2p + 1) = 1$.

For the edge $u_1v_i \in E(G)$, $\gcd(f(u_1), f(v_i)) = (p+2, f(v_i)) = 1$, since $p+2$ is a prime number and maximum value of $f(v_i)$ is p.
Thus f satisfy prime labelling and also it is possible to assign label 1 to any arbitrary vertex v_j, $where\ j = 1,2, \ldots\ldots m$.
Case 2: p be the lowest prime number in each pair of pseudo primes.
Subcase 1. If $v = v_j$ for some $j = 1,2,3 \ldots\ldots m$
Then define the labeling $f: V(G) \to \{1,2,3, \ldots\ldots 2p+1\}$ by

$$f(v_i) = p - j + i + 1\ for\ 1 \le i \le j-1$$
$$f(v_{j+i}) = i + 1, for\ 0 \le i \le p - j$$

For edges $\{v_iv_{i+1}\}, for\ 1 \le i \le p-1$, except $v_{j-1}v_j\ and\ v_jv_{j+1}$, $gcd(f(v_i), f(v_{i+1})) = 1$, since they are consecutive integers. For the edge $v_{j-1}v_j$, $\gcd\left(f(v_{j-1}), f(v_j)\right) = \gcd\left(f(v_{j-1}), 1\right) = 1$. Similarly we get a prime labelling for the edge v_jv_{j+1},

$f(u_1) = q$, where q be the pseudo prime of p.
$f(u_2) = q - 1$
$\vdots$
$f(u_{q-p}) = q - (q - p - 1) = p + 1$
$f(u_{q-p+1}) = 2p + 1$
$f(u_{q-p+2}) = 2p$
$\vdots$
$f(u_{p+1}) = q + 1$

For the edges $\{u_iu_{i+1}\}\ for\ 1 \le i \le p$, except $u_{q-p}u_{q-p+1}$, $\gcd(f(u_i), f(u_{i+1}) = 1$. Since the are consecutive integers.
For the edge $u_{q-p}u_{q-p+1}$, $\gcd(2p+1, p+1) = 1$
Consider the edge $u_1v_i\ for\ i = 1,2 \ldots\ldots p$. $\gcd(f(u_1), f(v_i)) = \gcd(q, f(v_i)) = 1$.
since q is a prime number greater than p and maximum value of $f(v_i)$ is p.
Thus f satisfy prime labelling and also it is possible to assign label 1 to any arbitrary vertex v_j, $where\ j = 1,2, \ldots\ldots m$.
Subcase 2. If $v = u_j$ for some $j = 1,2,3 \ldots\ldots n$
Then define the labeling $f: V(G) \to \{1,2,3, \ldots\ldots 2p+1\}$ by

$f(u_1) = q$
$f(v_i) = q - i, for\ 1 \le i \le p$

For the edge$\{v_i\ v_{i+1}\}\ for\ 1 \le i \le p-1, \gcd(f(v_i), f(v_{i+1})) = 1$, Since these are consecutive integers. Also for the edge $e = u_1v_i$, $\gcd(f(u_1), f(v_i)) = \gcd\left(q, f(v_i)\right) = 1$. Since q is a prime number and maximum value of $f(v_i)$ is $q-1$.

$$f(u_i) = \begin{cases} q + i - 1 & for\ 1 \le i < j \\ 1 & for\ j = 1 \\ q + i - 2 & for\ j + 1 \le i \le 2p - q + 3 \\ i - 2p + q - 2 & for\ 2p - q + 3 < i \le 1 \end{cases}$$

Then for any edge $\{u_iu_{i+1}\}, 1 \le i \le p+1$, except $u_{j-1}u_j$, $u_ju_{j+1}, u_{2p-q+3}u_{2p-q+4}$,
$(f(u_i), f(u_{i+1})) = 1$, since these are consecutive integers.
For the edge $u_{j-1}u_j$, $\left(f(u_{j-1}), f(u_j)\right) = \left(f(u_{j-1}), 1\right) = 1$. Similarly for the edge u_ju_{j+1},
For the edge $u_{2p-q+3}u_{2p-q+4}$, $\left(f(u_{2p-q+3}), f(u_{2p-q+4})\right) = (2p+1, 2) = 1$.
Thus f satisfy prime labelling and also it is possible to assign label 1 to any arbitrary vertex u_j, $where\ j = 1,2, \ldots\ldots n$.
Hence G is a strongly prime graph.
Theorem 2: The flower pot graph is a strongly prime graph.
Proof:
Let G be the flower pot graph with vertices, $\{u_1, u_2, u_3, \ldots\ldots u_m, u_{m+1}, u_{m+2}, \ldots\ldots\ldots u_{m+n}\}$. Then the edge set $E(G) = \{u_iu_{i+1}\ ; 1 \le i \le m-1\} \cup \{u_mu_1\} \cup \{u_1u_j\ ;\ m+1 \le j \le m+n\}$.
Let k be the index such that $f(u_k) = 1$.
Case.1: Let $k = 1$, that is $f(u_1) = 1$
Then we define a labeling $f: V \to \{1,2,3 \ldots\ldots m+n\}$ as follows.
$f(u_i) = i \quad for\ 1 \le i \le m+n$

Then for any edge $u_iu_{i+1} \in G, \gcd(f(u_i), f(u_{i+1})) = \gcd(i, i+1) = 1 \; for \; 1 \le i \le m-1$, since they are consecutive positive integers. For the edge $u_1u_m, \gcd(f(u_1), f(u_m)) = \gcd(1, f(u_m)) = 1$.

For the edge $\{u_1u_j\} \; for \; m+1 \le j \le m+n$, $\gcd\left(f(u_1), f(u_j)\right) = \gcd\left(1, f(u_j)\right) = 1$.

Case.2: If $f(u_i) = 2$, for some $i > k$.

Then we define a labeling $f: V \to \{1,2,3 \ldots \ldots . m+n\}$ as follows.

q be the largest prime number less than $m+n$

$$f(u_i) = \begin{cases} q-i+1 & if \; 1 \le i \le k-1 \\ 1 & if \; i = k \\ q-i+2 & if \; k+1 \le i \le q \end{cases}$$

If $m+n$ is even

$$f(u_{q+i}) = \begin{cases} m+n-i & for \; 1 \le i < m+n-q \\ m+n & for \quad i = m+n-q \end{cases}$$

If $m+n$ is odd

$$f(u_{q+i}) = m+n-i+1 \qquad for \; 1 \le i \le m+n-q$$

For edges $\{u_iu_{i+1}\}, for \; 1 \le i \le q-1$, except $u_{k-1}u_k \; and \; u_ku_{k+1,}$, $(f(u_i), f(u_{i+1})) = 1$, since they are consecutive integers. For the edge $u_{k-1}u_k$, $\left(f(u_{k-1}), f(u_k)\right) = (f(u_{k-1}), 1) = 1$. Similarly we get a prime labelling for the edge $u_ku_{k+1,}$

Consider the edge u_qu_{q+1},

For $m+n$ is even $\left(f(u_q), f(u_{q+1})\right) = (2, m+n-1) = 1$, since $m+n \; is \; even$.

For $m+n$ is odd $(\left(f(u_q), f(u_{q+1})\right) = (2, m+n) = 1$

For all other edges $\{u_{q+i}u_{q+i+1}\} for \; 1 \le i < m-q-1$, $\left(\left(f(u_{q+i}), f(u_{q+i+1})\right)\right) = 1$, since they are consecutive integers.

u_3 (3)

For the edge u_mu_1, $\left(f(u_m), f(u_1)\right) = (f(u_m), q) = 1$. Since q is the larg[illegible] number less than $m+n$, by Bertrand's postulate, $\frac{m+n}{2} < q < m+n$. which implies $m+n < 2q < 2(m+n)$.

Hence $2q \notin V(G)$. Therefore all numbers in the set $\{1,2,3, \ldots \ldots \ldots . m+n\}$ are relatively prime to q.

For all remaining edges $\{u_1u_j\} \; for \; m+1 \le j \le m+n$, $\gcd\left(f(u_1), f(u_j)\right) = \gcd\left(q, f(u_j)\right) = 1$. Since q is relatively prime to all numbers in $V(G)$.

Thus f is a prime labelling and also it is possible to assign label 1 to any arbitrary vertex u_i of G.

Case.3: If $f(u_i) = 2$, for some $i < k$.

Then we define a labeling $f: V \to \{1,2,3 \ldots \ldots . m+n\}$ as follows.

$f(u_i) = q-i+1 \; for \; 1 \le i \le q-1$

If $m+n$ is odd

$$f(u_{q+i}) = \begin{cases} m+n-i & for \; 0 \le i \le k-q-1 \\ 1 & for \; i = k-q \\ m+n+1-i & for \; k-q+1 \le i \le m+n \end{cases}$$

If $m+n$ is even

$$f(u_{q+i}) = \begin{cases} m+n-i-1 & for \; 0 \le i \le k-q-1 \\ 1 & for \; i = k-q \\ m+n-i & for \; k-q+1 \le i \le m+n-q \\ m+n & for \; i = m+n-q \end{cases}$$

For edges $\{u_iu_{i+1}\}, for \; 1 \le i \le q-2, gcd(f(u_i), f(u_{i+1})) = 1$, since they are consecutive integers.

Consider the edge $u_{q-1}u_q$,

If $m+n$ is even, $\gcd\left(f(u_{q-1}), f(u_q)\right) = (2, m+n-1) = 1$

If $m+n$ is odd, $\gcd\left(f(u_{q-1}), f(u_q)\right) = (2, m+n) = 1$

Suppose $f(u_k) = 1$ for some $k \le m$, then for the edge $u_{k-1}u_k$, $\left(f(u_{k-1}), f(u_k)\right) = (f(u_{k-1}), 1) = 1$. Similarly, we get a prime labelling for the edge $u_ku_{k+1,}$

For all other edges $\{u_{q+i}u_{q+i+1}\} for \; 0 \le i < m-q-1$, $\left(f(u_{q+i}), f(u_{q+i+1})\right) = 1$, since they are consecutive integers.

If $f(u_k) = 1$ for some $k > m$, then for the edge u_1u_k, $\left(f(u_1), f(u_k)\right) = (q, 1) = 1$.
For all the remaining edges,$\{u_1u_j;\ m \leq j \leq m+n\}$, $\left(f(u_1), f(u_j)\right) = \left(f(u_1), f(u_m)\right) = 1$. Since $f(u_1) = q$ and q is relatively prime to all numbers in $V(G)$.
Thus f is a prime labelling and also it is possible to assign label 1 to any arbitrary vertex u_i of G.
Hence G is a strongly prime graph.

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