HEAT FLOW OF ROD CONTROLLED BY DELAY DIFFERENCE OPERATOR

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Abstract: In this paper, we investigate the partial x-difference equation and arrive a model of it in discrete heat equation for rod using x-difference operators with shift values. Thus a corresponding model is the following discrete heat equation with

delay $\Delta_{(0,A_2)} v(k_1, k_2) = \gamma \Delta_{(\pm A_1,0)} v(k_1, k_2/\sigma)$, where σ is the delay time. The diffusion of heat is studied by the application of Fourier law of cooling and several solutions are provided for the same. Through numerical simulations using MATLAB, solutions are validated and applications are derived.

Keywords: partial difference equation, partial difference operator and discrete heat equation.

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1. INTRODUCTION

Difference operator Δ_α defined as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$ had its inception in 1984 by Jerzy Popenda. It was further extended by M.Maria Susai Manuel, et.al, [6] and G. Britto Antony Xavier, et.al,[3] by defining $\Delta_{\alpha(A)}$ as $\Delta_{\alpha(A)} v(k) = v(k A) - \alpha v(k)$ and $\Delta_{k(A)}$ as $\Delta_{k(A)} v(k) = v(k A) - k v(k)$ respectively. Here, we extend the theory of α-difference operator Δ_α to x-difference operator $\Delta_{x(A)}$, where $x = (x_1, x_2, \ldots, x_n)$ and $A = (A_1, A_2, \ldots, A_n)$ and obtain heat equation model by this x-difference operator. Partial difference and differential equations play a vital role in heat equations.

2. x-DIFFERENCE OPERATOR ON r-VARIABLES WITH SHIFT VALUES

For $x = (x_1, x_2, x_3, \ldots, x_r)$, the x-difference operator on r-variables real valued function with shift values $A = (A_1, A_2)$ is defined as,

$$\underset{x(A)}{\Delta} v(k_1, k_2) = v(k_1, k_2) - x_1 v\left(\frac{k_1}{A_1}, \frac{k_2}{A_2}\right) - x_2 v\left(\frac{k_1}{2A_1}, \frac{k_2}{2A_2}\right)$$

$$- x_3 v\left(\frac{k_1}{3A_1}, \frac{k_2}{3A_2}\right) - \ldots - x_r v\left(\frac{k_1}{rA_1}, \frac{k_2}{rA_2}\right) \qquad (1)$$

The operator in (1) becomes partial x-difference operator if either A_1 or A_2 is zero but not both. The equations involving a first order linear partial x-difference equation is,

$$\underset{x(A)}{\Delta} v(k) = u(k),\ A = (0, A_2) \quad \text{or} \quad (A_1, 0); \quad x = (x_1, x_2, \ldots, x_r) \qquad (2)$$

The equation (2) has a numerical solution of the form

$$v(k_1, k_2) - \sum_{i=0}^{m} \sum_{j=0}^{m} x_j F_{n-i-j} v\left(k_1, \frac{k_2}{(n+i)A_2}\right) = \sum_{i=0}^{n} F_i u\left(k_1, \frac{k_2}{iA_2}\right) \qquad (3)$$

where $F_0 = 1$ and $F_n = 0$ when $n < 0$.

3. DISCRETE HEAT EQUATION of A LONG ROD

Let $v(k_1, k_2)$ be the temperature at the position k_1 and time k_2 of long rod [3].

By (1) and Newton law of cooling, discrete heat equation of rod is expressed as

$$\underset{x(0,A_2)}{\Delta} v(k_1, k_2) = \gamma \underset{x(\pm A_1,0)}{\Delta} v(k_1, k_2/\sigma); \qquad x = (x_1, x_2, \ldots, x_r) \qquad (4)$$

where $\underset{x(\pm A_1,0)}{\Delta} = \underset{x(A_1,0)}{\Delta} + \underset{x(-A_1,0)}{\Delta}$. Our main aim of this paper is to study and discuss the solution of the heat equation (4). Here, we derive the temperature formula for $v(k_1, k_2)$ at the general position (k_1, k_2).

Theorem 3.1. *If* $\underset{x(\pm A_1)}{\Delta} v(k_1, k_2/\sigma) = u_{x(\pm A_1)}(k_1, k_2/\sigma)$ *are known, then the heat equation* (4) *has a solution*

$$v(k_1, k_2) = \sum_{i=0}^{m} \sum_{j=1}^{m} x_j F_{n+i-j} v(k_1, k_2/(n+i)A_2) + \gamma \sum_{i=0}^{n} F_i u(k_1, k_2/iA_2\sigma) \qquad (5)$$

Proof. By representing $\underset{x(\pm A_1)}{\Delta} v(k_1, k_2/\sigma) = u_{x(\pm A_1)}(k_1, k_2/\sigma)$, (4) becomes

$$v(k_1, k_2) = \sum_{i=0}^{m} \sum_{j=1}^{m} x_j F_{n+i-j} v(k_1, k_2/(n+i)A_2) + \gamma \underset{x(0,A_2)}{\Delta^{-1}} u_{x(\pm A_1)}(k_1, k_2/\sigma) \qquad (6)$$

The proof of (5) follows from the relation,

$\Delta_{x(0,A_2)}^{-1} u_{x(\pm A_1)}(k_1, k_2) = \sum_{i=1}^{n} F_i - u_{(\pm A_1)} (k_1 \pm r(0), k_2 \quad iA_2)$ and (6).

Theorem 3.2. *Consider* (4) *and denote* $v(k_1 \pm A_1, *) = v(k_1 \; A_1, *) + v(k_1 / A_1, *)$ *and* $v(k_1 \; 2A_1, *) = v(k_1 \; 2A_1, *) + v(k_1 / 2A_1, *)$*. Then, the following four types solutions of the equation* (4) *are equivalent*

a)

$$v(k_1,k_2)v\left(k_1,\frac{k_2}{A_{2m}}\right) - \sum_{i=1}^{m}\gamma x_1^i v\left(k_1A_1,\frac{k_2}{A_2^{i-1}}\right) + \sum_{i=1}^{m}\gamma x_1^{i-1} v\left(k_1A_1,\frac{k_2}{A_2^{i-1}}\right) + \sum_{r=2}^{n}\left\{\sum_{i=1}^{n} x_r x_1^{i-1} v\left(k_1,\frac{k_2}{A_{2m}}\right) - \gamma v(k_1A_1^r,\frac{k_2}{A_{2(i-1)\sigma}})\right\} \quad (7)$$

b)

$$v(k_1,k_2) = x_1^m v(k_1,k_2A_{2^m}) - \sum_{i=1}^{m}\frac{\gamma}{x_1^i} v\left(k_1A_1,\frac{k_2}{A_2^{i-1}}\right) + \sum_{i=1}^{m}\frac{\gamma}{x_1^{i-1}} v\left(k_1A_1,\frac{k_2}{A_2^{i-1}}\right) + \sum_{r=2}^{n}\left\{\sum_{i=1}^{n}\frac{x_r}{x_1^{i-1}} v\left(k_1,\frac{k_2}{A_{2m}}\right) - \gamma v(k_1A_1^r,\frac{k_2}{A_{2(i-1)\sigma}})\right\} \quad (8)$$

c)

$$v(k_1,k_2) = \frac{1}{\gamma^m} v\left(\frac{k_1}{A_1^m},\frac{k_2\sigma^m}{A_2^m}\right) - \sum_{i=1}^{m}\frac{1}{x_1\gamma^i} v\left(\frac{k_1}{A_1^i},\frac{k_2\sigma^i}{A_2^{i-1}}\right) - \sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v\left(\frac{k_1}{A_1^{i+1}},\frac{k_2\sigma^{i-1}}{A_2^{i-1}}\right) + \sum_{i=1x_1}^{m}\frac{1}{\gamma^{i-1}} v\left(\frac{k_1}{A_1^i},\frac{k_2\sigma^{i-1}}{A_2^{i-1}}\right) - \sum_{r=2}^{m}\left\{\sum_{i=1}^{n}\frac{x_r}{x_1x_1^{i-1}} v\left(\frac{k_1}{A_1^{i-r}},\frac{k_2}{A_{2m}}\right) + v(\frac{k_1}{A_1^{r+i}},\frac{k_2}{A_{2(i-1)\sigma^{i-1}}})\right\} + \sum_{r=2}^{n}\left\{\sum_{i=1}^{n}\frac{x_r}{x_1x_1^{i-1}} v\left(\frac{k_1}{A_1^i},\frac{k_2}{A_{2^{i+(r-1)}}}\right)\right\} \quad (9)$$

d)

$$v(k_1,k_2) = \frac{1}{\gamma^m} v(k_1A_1^m,k_2\sigma^m A_2^m) - \sum_{i=1}^{m}\frac{1}{x_1\gamma^i} v\left(k_1A_1^i,k_2\sigma^i A_2^{i-1}\right) - \sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v\left(k_1A_1^{i+1},k_2\sigma^{i-1}A_2^{i-1}\right) + \sum_{i=1x_1}^{m}\frac{1}{\gamma^{i-1}} v\left(k_1A_1^i,k_2\sigma^{i-1}A_2^{i-1}\right) - \sum_{r=2}^{m}\left\{\sum_{i=1}^{n}\frac{x_r}{x_1x_1^{i-1}} v\left(k_1A_1^{i-r},k_2A_{2^m}\right) + v(k_1A_1^{r+i},k_{2A_{2(i-1)\sigma^{i-1}}})\right\} + \sum_{r=2}^{n}\left\{\sum_{i=1}^{n}\frac{x_r}{x_1x_1^{i-1}} v\left(k_1A_1^i,k_2A_{2^{i+(r-1)}}\right)\right\} \quad (10)$$

Proof. (a). From (4), directly generates the relation

$$v(k_1,k_2) = x_1 v\left(k_1,\frac{k_2}{A_2}\right) + \cdots . + x_r v\left(k_1,\frac{k_2}{A_r}\right) - x_1\gamma v(k_1A_1,k_2) - \cdots \ldots - x_r\gamma v(k_1A_1^r,k_2) \tag{11}$$

By replacing k_2 by k_2 / A_2, $k_2 / 2A_2$, ..., k_2/ mA_2 in (11), we obtain expressions for $v(k_1, k_2 - rA_2)$ and $v(k_1 \pm rA_1, k_2)$. Now proof of (a) follows by applying all these values in (11).

(b) The heat equation (4), we arrive the relation

$$v(k_1,k_2) = \frac{1}{x_1}v(k_1,k_2A_2) + \gamma v\left(k_1A_1,\frac{k_2}{A_2}\sigma\right) - \frac{\gamma}{x_1}v\left(k_1,\frac{k_2}{A_{2\sigma}}\right) - \frac{x_2}{x_1}\left(v\left(k_1,\frac{k_2}{A_2}\right) - \gamma v\left(k_1A_1^2,\frac{k_2}{A_2}\right)\right)$$
$$-\frac{x_{r-1}}{x_r}\left(v\left(k_1,\frac{k_2}{A_2}\right) - \gamma v(k_1A_1^{r-1},k_2A_2)\right) - \frac{x_r}{x_1}(v\left(k_1,\frac{k_2}{A_2}\right) - \gamma v(k_1A_1^r,\frac{k_2}{A_2^\sigma})) \tag{12}$$

The proof of (b) follows by replacing k_2 by $k_2\, A_2$, $k_2\, 2A_2$, ..., $k_2\, mA_2$ repeatedly and substituting corresponding v-values in (12).

(c). A simple calculation on (4) gives the expression

$$(k_1,k_2) = \frac{1}{\gamma}v\left(\frac{k_1}{A_1},\frac{k_2\sigma}{A_2}\right) - v\left(\frac{k_1}{A_1^2},k_2\right) + \frac{1}{x_1}v\left(\frac{k_1}{A_1},k_2\right), \frac{x_2}{x_{1\gamma}}v\left(\frac{k_1}{A_1},\frac{k_{2\sigma}}{A_2^2}\right) + \frac{x_3}{x_{1\gamma}}v\left(\frac{k_1}{A_1},\frac{k_{2\sigma}}{a_2^3}\right) + \cdots +$$
$$\frac{x_r}{x_{1\gamma}}v\left(\frac{k_1}{A_1},\frac{k_2\sigma}{A_2^r}\right) - \frac{x_2}{x_1}\left(v(k_1A_{1,k_2}) - v\left(\frac{k_1}{A_1^3},k_2\right)\right) - \cdots . - \frac{x_r}{x_1}\left(v(k_1A_1^{r-1},k_2) + v\left(\frac{k_1}{A_1^{r+1}},k_2\right)\right) -$$
$$\frac{1}{x_{1\gamma}}v(\frac{k_1}{a_1},k_2\sigma) \tag{13}$$

The proof of (c) follows by replacing k_1 by $k_1 - A_1$, $k_1 - 2A_1$, ..., $k_1 - mA_1$ and k_2 by k_2 / A_2, $k_2 / 2A_2$, ..., k_2/ mA_2 repeatedly.

For numerical verification, we give the MATLAB coding for (a) of Theorem (3.2) *when* $m = 1$, $r = 2$, $k_1 = 1$, $A_1 = 1$, $k_2 = 2$, $A_2 = 2$, $x_1 = 1$, $x_2 = 2$, $v(k_1, k_2) = e^{(k_1+k_2)}$ *and* γ *is as given in* (13).

For numerical verification, if we assume that $k_1 = 1$, $k_2 = 2$, $A_1 = 2$, $A_2 = 3$, $m = 2$

$$\frac{a^3 - 1}{}$$

then $v(k_1, k_2) = a^3$, $\gamma = a^2 + a^{-2} - 2$
, LHS and RHS of (a), (b) of Theorem (3.2)
are given below respectively.

$$20.08 = 34.9228(0.0498) + 18.3473 \Rightarrow 20.08 = 20.08.$$

4. CONCLUSION

The newly developed partial difference operator generates many applications in the field of finite difference methods and heat equation. The nature of propagation of heat through materials are derived using partial *x*-difference operator. The core Theorem provide the possibility of predicting the temperature with some delay either for the past or the future after getting the temperature at few finite points.

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