Sigma coloring of some cycle related graphs

J. Suresh Kumar[1] and Preethi K Pillai[2]

Post Graduate and Research Department of Mathematics,

N.S.S. Hindu College, Changanacherry, Kerala, India-686102.

Abstract: In this paper, we investigate the σ-coloring and obtain the Sigma Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph.

Key Words: Graph, Sigma coloring, Sigma Chromatic number, Wheel graph, Helm graph, Gear graph, Flower graph, Friendship graph, Crown graph, Web graph.

1. Introduction

Graph coloring take a major stage in Graph Theory since the advent of the famous four color conjecture. Several variations of graph coloring were investigated [2] and new types of coloring are still available recently [5, 6, 7]. The σ −coloring (Sigma coloring) was introduced by Gary Chartrand et.al.[3]. By a graph, we mean a finite undirected graph without loops or parallel edges. For the terms and notations not defined explicitly here, reader may refer Harary [9].

We begin by recalling some basic definitions which are useful for the present investigation.

Definition.1.1. The Wheel graph, $W_n, n \geq 3$, is the join of the graphs C_n and K_1 .That is, W_n is the (n+1)-vertex graph obtained from the graph C_n by adding a new vertex, v and joining it to each of the n vertices of the cycle, C_n. Here we call the vertices corresponding to C_n as rim vertices and the vertex corresponding to K_1 (the newly added vertex) is called the apex vertex.

Definition 1.2. The Helm graph $H_n, n \geq 3$ is the graph obtained from Wheel graph, W_n by adding a pendent edge at each vertex on the rim of the Wheel, W_n.

Definition 1.3. The closed Helm graph, CH_n, is the graph obtained from a Helm graph H_n and adding edges between the pendent vertices.

Definition 1.4. The Gear graph, G_n , is a graph obtained from Wheel graph, W_n by adding an extra vertex between each pair of adjacent vertices on the rim of the Wheel graph W_n.

Definition 1.5. Flower graph FL_n is the graph obtained from a Helm graph by joining each pendant vertex to the central vertex of the Helm.

Definition 1.6. The Friendship graph, F_n can be constructed by joining n copies of the cycle Graph, C_3 to a common vertex.

Definition 1.7. The Double Wheel graph, DW_n of size n is composed of $2C_n + K_1$. It consists of two cycles C_n, where vertices of each of these two cycles are connected to a common vertex.

Definition 1.8. The Crown graph, C_n^+ is obtained from the cycle graph, C_n by adding a pendent edge to each vertex of C_n

Definition1.9. The Double crown graph, C_n^{++} is the graph obtained from the cycle C_n by adding two pendent edge at each vertex of C_n

Definition1.10. The Web graph is the graph obtained from a Helm graph by joining the pendent vertices of the Helm to form a cycle and then adding a pendent edge to each vertex of the outer cycle.

Definition1.11 The floor function of a real number x is the largest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$. The ceil function of a real number x is the smallest integer greater than or equal to x and it is denoted by $\{x\}$.

Definition1.12. Let G be a simple connected graph and C: $V(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, be a coloring of the vertices in G. For any v$\in V(G)$, let $\sigma(v)$ denotes the sum of colors of the vertices adjacent to v then f is called a Sigma coloring ($\sigma -$ coloring) of G if for any two adjacent vertices $u, v \in V(G), \sigma(v) \neq \sigma$(u). The minimum Number of colors used in a sigma coloring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$.

2. Main Results:

In this section, we discuss the Sigma Coloring of the cycle related Graphs mentioned above. For the terms and definitions not explicitly defined here, reader may refer Harary [9].

Theorem.2.1. The Wheel graph W_n is σ-colorable and its Sigma chromatic number is given by

$$\sigma(W_n) = \begin{cases} 4 & if\ n = 3 \\ 2 & if\ n \geq 4\ and\ even \\ 3 & if\ n \geq 4\ and\ is\ odd \end{cases}$$

Proof.

Case.1. $n = 3$

Let the central vertex of the Wheel graph, W_n be v and the vertices on the rim are v_1 ,v_2,v_3. Color the vertex v with color 1 and the vertices v_1, v_2, v_3 with the colors $2, 3, 4$ respectively so that the sum of colors of adjacent vertices is different. The coloring is σ –colorable and since the vertices v_1, v_2, v_3 are mutually adjacent, at least 4 colors are needed in any coloring of W_n so that $\sigma(W_3)$ =4

Case.2. $n \geq 4$ and is even

Let the central vertex of the Wheel graph, W_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$

Define a coloring function C : V(W_n) → {1,2} as follows.

$C(v) = 1$

$C(v_{2i}) = 2$ if $1 \leq i \leq \frac{n}{2}$

$C(v_{2i-1}) = 1$ if $1 \leq i \leq \frac{n}{2}$

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(W_n) =2.

Case.3. $n \geq 4$ and is odd

Let the central vertex of the Wheel graph, W_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$

Define a coloring function C : V(W_n)→ {1,2,3} as follows.

$C(v) = 1$

C(v_{3i})= 2 if $1 \leq i \leq \left\lfloor\frac{n}{3}\right\rfloor$

C(v_{3i-1})= 1 if $1 \leq i \leq \left\lfloor\frac{n+1}{3}\right\rfloor$

C(v_{3i-2})= 3 if $1 \leq i \leq \left\lfloor\frac{n+2}{3}\right\rfloor$

Then this coloring is a minimal $\sigma -$ coloring using only 3 colors. So σ(W_n) =3.

Theorem. 2.2. The Helm graph H_n is σ-colorable for n≥ 4 and Sigma chromatic number is given by σ(H_n) =2.

Proof: Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

Case.1. n≥ 4 and n is even

Define C : V(H_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if $1 \leq i \leq \frac{n}{2}$

C(v_{2i-1})= 1 if $1 \leq i \leq \frac{n}{2}$

C(w_{2i})= 1 if $1 \leq i \leq \frac{n}{2}$

C(w_{2i-1})= 2 if $1 \leq i \leq \frac{n}{2}$

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(H_n) =2.

Case.2. n≥ 4 and n is odd

Define C : V(H_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if $1 \leq i \leq \frac{n-1}{2}$

C(v_{2i-1})= 1 if $1 \leq i \leq \frac{n+1}{2}$.

C(w_i)= 1 if $1 \leq i \leq n$-1

C(w_n)= 2.

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(H_n) =2.

Theorem. 2.3. The Closed Helm graph, CH_n is σ-colorable for n≥ 4 and σ(CH_n) =2.

Proof: Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

Case.1. n≥ 4 and n is even

Define C : V(CH_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \le \frac{n}{2}$

C(v_{2i-1})= 1 if 1≤ $i \le \frac{n}{2}$.

C(w_{2i})= 1 if 1≤ $i \le \frac{n}{2}$

C(w_{2i-1})= 2 if 1≤ $i \le \frac{n}{2}$.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(CH_n) =2.

Case.2. $n = 5$

Define C : V(CH_n)→ {1,2} as follows:

$C(v) = 2$

C(v_1)=2, C(v_2)=2, C(v_4)=2, C(v_5)=2 and C(v_3)=1.

C(w_i)= 1 if 1≤ $i \le 4$, C(w_5)= 2.

Then this coloring satisfies all the conditions of $\sigma -$ coloring using only 2 colors. So σ(CH_n) =2.

Case.3. n≥ 3 and n is odd

Define C : V(CH_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \le \frac{n-1}{2}$

C(v_{2i-1})= 1 if 1≤ $i \le \frac{n+1}{2}$.

C(w_{2i})= 1 if 1≤ $i \le \frac{n-1}{2}$

C(w_{2i-1})= 2 if 1≤ $i \le \frac{n-1}{2}$

C(w_n)= 1.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(CH_n) =2.

Theorem. 2.4. The Gear graph, G_n is σ-colorable and σ(G_n) =2.

Proof: Let the central vertex of the Gear graph, G_n be v and the vertices on the rim are $v_1, v_2, \dots v_n$ and the newly added vertices are $v_1{}'$,$v_2{}'$,$v_3{}'$,,…………$v_n{}'$.

Case.1. n≥ 3 and n is even

Define C : V(G_n)→ {1,2} as follows:

$C(v) = 2$

C(v_{2i})= 2 if 1≤ $i \le \frac{n}{2}$

C(v_{2i-1})= 1 if 1≤ $i \le \frac{n}{2}$.

C($v_j{}'$)= 1 if 1≤ $j \le n$

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(G_n) =2

Case.2. n≥ 3 and n is odd

Define C : V(G_n)→ {1,2}as follows:

$C(v) = 2$

C(v_{2i})= 2 if 1≤ $i \le \frac{n-1}{2}$

C(v_{2i-1})= 1 if 1≤ $i \le \frac{n+1}{2}$.

C($v_j{}'$)= 1 if 1≤ $j \le n$.

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(G_n) =2

Theorem. 2.5. The Flower graph FL_n is σ-colorable and σ(FL_n) =2

Proof: Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \dots v_n$ and the pendent vertices corresponding to the cycle are $w_{1,}w_2$,w_3…………w_n.

Case.1. n≥ 3 and n is even

Define C : V(FL_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \le \frac{n}{2}$

C(v_{2i-1})= 1 if 1≤ $i \le \frac{n}{2}$.

C(w_i)= 1 if 1≤ $i \le n$.

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(FL_n) =2

Case.2. n> 3 and n is odd

Define C : V(FL_n)→ {1,2} as follows.

C(v)=1

C(v_{2i})= 2 if 1≤ $i \leq \frac{n-1}{2}$.

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n+1}{2}$.

C(w_i)= 1 if $2 \leq i \leq n$.

C(w_1)= 2 .

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(FL_n) =2

Theorem. 2.6. The Friendship graph F_n is σ-colorable and σ(F_n) =2 .

Proof: Let the central vertex of the Friendship graph F_n be v and let $v_{11,}v_{12}$ be the vertices of the first copy of C_3, $v_{21,}v_{22}$ be the vertices of the second copy of C_3, $v_{31,}v_{32}$ be the vertices of the third copy of C_3 and so on. Let $v_{n1,}v_{n2}$ be the vertices of the nth copy of C_3.

Define C : V(F_n)→ {1,2} as follows.

$C(v) = 1$

C(v_{i1})= 1 if 1≤ $i \leq n$.

C(v_{i2})= 2 if 1≤ $i \leq n$.

Then this coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(F_n) =2

Theorem.2.7. The Double Wheel graph, DW_n is σ-colorable and σ(DW_n) $=\begin{cases} 2 \ if \ n \ is \ even \\ 3 \ if \ n \ is \ odd \end{cases}$

Proof : Let v be the apex vertex. Let $\{v_{1,}v_2, v_3 \dots \dots \dots \dots . . v_n\}$ and $\{u_{1,}u_2, u_3 \dots \dots \dots \dots . . u_n\}$ be vertices of inner and outer cycles of C_n respectively.

Case.1. n> 3 and n is even

Let v be the central vertex.

Define C : V(DW_n)→ {1,2} as follows.

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$.

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

C(u_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$.

C(u_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(DW_n) =2

Case.2. n is odd

Define C : V(DW_n)→ {1,2,3} as follows.

$C(v) = 1$

C(v_{3i})= 2 if 1≤ $i \leq \left\lfloor \frac{n}{3} \right\rfloor$

C(v_{3i-1})= 1 if 1≤ $i \leq \left\lfloor \frac{n+1}{3} \right\rfloor$

C(v_{3i-2})= 3 if 1≤ $i \leq \left\lfloor \frac{n+2}{3} \right\rfloor$

C(u_{3i})= 2 if 1≤ $i \leq \left\lfloor \frac{n}{3} \right\rfloor$

C(u_{3i-1})= 1 if 1≤ $i \leq \left\lfloor \frac{n+1}{3} \right\rfloor$

C(u_{3i-2})= 3 if 1≤ $i \leq \left\lfloor \frac{n+2}{3} \right\rfloor$

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(DW_n) =3

Theorem. 2.8. The Crown graph C_n^+ is σ-colorable and σ(C_n^+) =2

Proof: Let the vertices on the cycle be $v_{1,}v_2,v_3 \dots \dots \dots \dots . v_n$ and the pendent vertices corresponding to the cycle be $w_{1,}w_2,w_3 \dots \dots \dots \dots w_n$.

Case.1. n> 3 and n is even

Define C : V(C_n^+)→ {1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$.

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

C(w_i)= 1 if 1≤ $i \leq n$.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(C_n^+) =2

Case.2. n> 3 and n is odd

Define C : VC_n^+)→ {1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n-1}{2}$.

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n+1}{2}$.

C(w_i)= 1 if $1 \leq i \leq n-1$.

C(w_n)= 2 .

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(C_n^+) =2

Theorem. 2.9. The Double Crown graph, C_n^{++}, is σ-colorable and σ(C_n^{++}) =2 .

Proof: Let $\{v_{1,}v_2, v_3, \ldots .. v_n\}$ be the vertices of the cycle C_n. Let $\{v_{i1,}v_{i2}\}$ be the pendent edge corresponding to each vertex, $v_i, 1 \leq i \leq n$.

Case.1. n > 3 and n is even

Define C : V(C_n^{++})→ {1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$; C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

C(v_{i1})= 1 = C(v_{i2}) if 1≤ $i \leq n$.

Case.2. n> 3 and n is odd

Define C : V(C_n^{++})→ {1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n-1}{2}$; C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n+1}{2}$.

C(v_{i1})= 1 if 1≤ $i \leq n$; C(v_{i2})=1 if 1≤ $i \leq n-1$; C(v_{n2})=2

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(C_n^{++}) =2

Theorem. 2.10. The Web graph, Wb_n is σ-colorable and σ(Wb_n) =2.

Proof: Let the central vertex of the Web graph Wb_n be v. Let the vertices on the innercycle be $v_{1,}v_2, v_3, \ldots, v_n$ and the vertices on the outer cycle be $u_{1,}u_2, u_3 \ldots \ldots \ldots \ldots . u_n$ and the pendent vertices be $w_{1,}w_2$,w_3...........w_n.

Case.1. $n > 3$ and n is even

Define C : V(Wb_n)→ {1,2} as follows:

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$; C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

C(u_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$; C(u_{2i-1})= 2 if 1≤ $i \leq \frac{n}{2}$.

C(w_i)= 1, if 1≤ $i \leq n$.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(Wb_n) =2

Case.2. $n = 5$

Define C : V(Wb_n)→ {1,2} as follows:

$C(v) = 2$

C(v_1)=2, C(v_2)=2, $C(v_3) = 1$, C(v_4)=2, C(v_5)=2

C(u_1)=1, C(u_2)=1, C(u_3)=1, C(u_4)=1, $C(u_5) = 2$

C(w_i)= 1 if 1≤ $i \leq 3$, C(w_4)= 2, C(w_5)= 1

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(Wb_n) =2

Case.3. $n > 5$ and n is odd

Define C : V(Wb_n)→ {1,2} as follows.

$C(v) = 1$

C(v_{2i})= 2 if 1≤ $i \leq \frac{n-1}{2}$; C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n+1}{2}$.

C(u_{2i})= 1, if 1≤ $i \leq \frac{n-1}{2}$; C(u_{2i-1})= 2, if 1≤ $i \leq \frac{n-1}{2}$; C(u_n)= 1.

C(w_i)= 1 if 1≤ $i \leq n-1$; C(w_n)= 2.

This coloring is a minimal $\sigma -$ coloring using only 2 colors. So σ(Wb_n) =2

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