Sigma coloring of some cycle related graphs

J. Suresh Kumar¹ and Preethi K Pillai²

Post Graduate and Research Department of Mathematics,
N.S.S. Hindu College, Changanacherry, Kerala, India-686102.

Abstract: In this paper, we investigate the σ-coloring and obtain the Sigma Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph.

Key Words: Graph, Sigma coloring, Sigma Chromatic number, Wheel graph, Helm graph, Gear graph, Flower graph, Friendship graph, Crown graph, Web graph.

1. Introduction

Graph coloring take a major stage in Graph Theory since the advent of the famous four color conjecture. Several variations of graph coloring were investigated [2] and new types of coloring are still available recently [5, 6, 7]. The σ – coloring (Sigma coloring) was introduced by Gary Chartrand et.al.[3]. By a graph, we mean a finite undirected graph without loops or parallel edges. For the terms and notations not defined explicitly here, reader may refer Harary [9].

We begin by recalling some basic definitions which are useful for the present investigation.

Definition 1.1. The Wheel graph, \( W_n, n \geq 3 \), is the join of the graphs \( C_n \) and \( K_1 \). That is, \( W_n \) is the (n+1)-vertex graph obtained from the graph \( C_n \) by adding a new vertex, \( v \) and joining it to each of the n vertices of the cycle, \( C_n \). Here we call the vertices corresponding to \( C_n \) as rim vertices and the vertex corresponding to \( K_1 \) (the newly added vertex) is called the apex vertex.

Definition 1.2. The Helm graph \( H_n, n \geq 3 \), is the graph obtained from Wheel graph, \( W_n \) by adding a pendant edge at each vertex on the rim of the Wheel, \( W_n \).

Definition 1.3. The closed Helm graph, \( CH_n \), is the graph obtained from a Helm graph \( H_n \) and adding edges between the pendant vertices.

Definition 1.4. The Gear graph, \( G_n \), is a graph obtained from Wheel graph, \( W_n \) by adding an extra vertex between each pair of adjacent vertices on the rim of the Wheel graph \( W_n \).

Definition 1.5. Flower graph \( FL_n \) is the graph obtained from a Helm graph by joining each pendant vertex to the central vertex of the Helm.

Definition 1.6. The Friendship graph, \( F_n \) can be constructed by joining \( n \) copies of the cycle Graph, \( C_3 \) to a common vertex.

Definition 1.7. The Double Wheel graph, \( DW_n \) of size \( n \) is composed of \( 2C_n + K_1 \). It consists of two cycles \( C_n \), where vertices of each of these two cycles are connected to a common vertex.

Definition 1.8. The Crown graph, \( C_n^+ \) is obtained from the cycle graph, \( C_n \) by adding a pendant edge to each vertex of \( C_n \).

Definition 1.9. The Double crown graph, \( C_n^{++} \) is the graph obtained from the cycle \( C_n \) by adding two pendant edge at each vertex of \( C_n \).

Definition 1.10. The Web graph is the graph obtained from a Helm graph by joining the pendant vertices of the Helm to form a cycle and then adding a pendant edge to each vertex of the outer cycle.

Definition 1.11. The floor function of a real number \( x \) is the largest integer less than or equal to \( x \) and it is denoted by \( \lfloor x \rfloor \). The ceil function of a real number \( x \) is the smallest integer greater than or equal to \( x \) and it is denoted by \( \lceil x \rceil \).

Definition 1.12. Let \( G \) be a simple connected graph and \( \Sigma: V(G) \rightarrow \mathbb{N} \), where \( \mathbb{N} \) is the set of positive integers, be a coloring of the vertices in \( G \). For any \( v \in V(G) \), let \( \sigma(v) \) denotes the sum of colors of the vertices adjacent to \( v \) then \( f \) is called a Sigma coloring (\( \sigma \) – coloring ) of \( G \) if for any two adjacent vertices \( u, v \in V(G) \), \( \sigma(v) \neq \sigma(u) \). The minimum Number of colors used in a sigma coloring of \( G \) is called the sigma chromatic number of \( G \) and is denoted by \( \sigma(G) \).

2. Main Results:

In this section, we discuss the Sigma Coloring of the cycle related Graphs mentioned above. For the terms and definitions not explicitly defined here, reader may refer Harary [9].

Theorem 2.1. The Wheel graph \( W_n \) is \( \sigma \)-colorable and its Sigma chromatic number is given by
\[
\sigma(W_n) = \begin{cases} 
4 & \text{if } n = 3 \\
2 & \text{if } n \geq 4 \text{ and even} \\
3 & \text{if } n \geq 4 \text{ and is odd}
\end{cases}
\]

**Proof.**

**Case 1.** \( n = 3 \)

Let the central vertex of the Wheel graph, \( W_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, v_3 \). Color the vertex \( v \) with color 1 and the vertices \( v_1, v_2, v_3 \) with the colors 2, 3, 4 respectively so that the sum of colors of adjacent vertices is different. The coloring is \( \sigma \)–colorable and since the vertices \( v_1, v_2, v_3 \) are mutually adjacent, at least 4 colors are needed in any coloring of \( W_n \) so that \( \sigma(W_3) = 4 \)

**Case 2.** \( n \geq 4 \) and is even

Let the central vertex of the Wheel graph, \( W_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \)

Define a coloring function \( C : V(W_n) \to \{1, 2\} \) as follows.

\[
C(v) = 1
\]

\[
C(v_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n}{2}
\]

\[
C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}
\]

Then this coloring is a minimal \( \sigma \) – coloring using only 2 colors. So \( \sigma(W_n) = 2 \).

**Case 3.** \( n \geq 4 \) and is odd

Let the central vertex of the Wheel graph, \( W_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \)

Define a coloring function \( C : V(W_n) \to \{1, 2, 3\} \) as follows.

\[
C(v) = 1
\]

\[
C(v_{3i}) = 2 \text{ if } 1 \leq i \leq \frac{n}{3}
\]

\[
C(v_{3i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n+1}{3}
\]

\[
C(v_{3i-2}) = 3 \text{ if } 1 \leq i \leq \frac{n+2}{3}
\]

Then this coloring is a minimal \( \sigma \) – coloring using only 3 colors. So \( \sigma(W_n) = 3 \).

**Theorem 2.2.** The Helm graph \( H_n \) is \( \sigma \)–colorable for \( n \geq 4 \) and Sigma chromatic number is given by \( \sigma(H_n) = 2 \).

**Proof:** Let the central vertex of the Helm graph \( H_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \) and the pendant vertices are \( w_1, w_2, w_3, \ldots, w_n \).

**Case 1.** \( n \geq 4 \) and \( n \) is even

Define \( C : V(H_n) \to \{1, 2\} \) as follows:

\[
C(v) = 1
\]

\[
C(v_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n}{2}
\]

\[
C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}
\]

Then this coloring is a minimal \( \sigma \) – coloring using only 2 colors. So \( \sigma(H_n) = 2 \).

**Case 2.** \( n \geq 4 \) and \( n \) is odd

Define \( C : V(H_n) \to \{1, 2\} \) as follows:

\[
C(v) = 1
\]

\[
C(v_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2}
\]

\[
C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n+1}{2}
\]

\[
C(w_i) = 1 \text{ if } 1 \leq i \leq n-1
\]

\[
C(w_n) = 2
\]

Then this coloring is a minimal \( \sigma \) – coloring using only 2 colors. So \( \sigma(H_n) = 2 \).

**Theorem 2.3.** The Closed Helm graph, \( CH_n \) is \( \sigma \)–colorable for \( n \geq 4 \) and \( \sigma(CH_n) = 2 \).

**Proof:** Let the central vertex of the Helm graph \( H_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \) and the pendant vertices are \( w_1, w_2, w_3, \ldots, w_n \).

**Case 1.** \( n \geq 4 \) and \( n \) is even

Define \( C : V(CH_n) \to \{1, 2\} \) as follows:

\[
C(v) = 1
\]
Case 2. \( n = 5 \)
Define \( C : V(\text{CH}_5) \to \{1, 2\} \) as follows:
\[
C(v) = 2
\]
\[
C(v_1) = 2, C(v_2) = 2, C(v_4) = 2, C(v_5) = 2 \text{ and } C(v_3) = 1.
\]
\[
C(w) = 1 \text{ if } 1 \leq i \leq 4, \ C(w_2) = 2.
\]
Then this coloring satisfies all the conditions of \( \sigma \) — coloring using only 2 colors. So \( \sigma(\text{CH}_5) = 2 \).

Case 3. \( n \geq 3 \) and \( n \) is odd
Define \( C : V(\text{CH}_n) \to \{1, 2\} \) as follows:
\[
C(v) = 1
\]
\[
C(v_1) = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2},
\]
\[
C(v_2) = 1 \text{ if } 1 \leq i \leq \frac{n+1}{2}.
\]
\[
C(w) = 1 \text{ if } 1 \leq i \leq \frac{n-1}{2}
\]
\[
C(w_1) = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2}.
\]
\[
C(w_n) = 1.
\]
This coloring is a minimal \( \sigma \) — coloring using only 2 colors. So \( \sigma(\text{CH}_n) = 2 \).

Theorem 2.4. The Gear graph, \( G_n \) is \( \sigma \) — colorable and \( \sigma(G_n) = 2 \).

Proof: Let the central vertex of the Gear graph, \( G_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \) and the newly added vertices are \( v_1', v_2', v_3', \ldots, v_n' \).

Case 1. \( n \geq 3 \) and \( n \) is even

Define \( C : V(G_n) \to \{1, 2\} \) as follows:
\[
C(v) = 2
\]
\[
C(v_1) = 2 \text{ if } 1 \leq i \leq \frac{n}{2},
\]
\[
C(v_2) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}.
\]
\[
C(v_j) = 1 \text{ if } 1 \leq j \leq n.
\]
Then this coloring is a minimal \( \sigma \) — coloring using only 2 colors. So \( \sigma(G_n) = 2 \).

Case 2. \( n \geq 3 \) and \( n \) is odd

Define \( C : V(G_n) \to \{1, 2\} \) as follows:
\[
C(v) = 2
\]
\[
C(v_1) = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2},
\]
\[
C(v_2) = 1 \text{ if } 1 \leq i \leq \frac{n+1}{2}.
\]
\[
C(v_j) = 1 \text{ if } 1 \leq j \leq n.
\]
Then this coloring is a minimal \( \sigma \) — coloring using only 2 colors. So \( \sigma(G_n) = 2 \).

Theorem 2.5. The Flower graph \( FL_n \) is \( \sigma \) — colorable and \( \sigma(FL_n) = 2 \).

Proof: Let the central vertex of the Helm graph \( H_n \) be \( v \) and the vertices on the rim are \( v_1, v_2, \ldots, v_n \) and the pendent vertices corresponding to the cycle are \( w_1, w_2, w_3, \ldots, w_n \).

Case 1. \( n \geq 3 \) and \( n \) is even

Define \( C : V(FL_n) \to \{1, 2\} \) as follows:
\[
C(v) = 1
\]
\[
C(v_1) = 2 \text{ if } 1 \leq i \leq \frac{n}{2},
\]
\[
C(v_2) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}.
\]
\[
C(w_i) = 1 \text{ if } 1 \leq i \leq n.
\]
Then this coloring is a minimal \( \sigma \) — coloring using only 2 colors. So \( \sigma(FL_n) = 2 \).

Case 2. \( n > 3 \) and \( n \) is odd
Define $C : V(FL_n) \rightarrow \{1,2\}$ as follows.

$C(v) = 1$

$C(v_{2i}) = 2$ if $1 \leq i \leq \frac{n-1}{2}$.

$C(v_{2i-1}) = 1$ if $1 \leq i \leq \frac{n+1}{2}$.

$C(w_i) = 1$ if $2 \leq i \leq n$.

$C(w_i) = 2$.

Then this coloring is a minimal $\sigma$ – coloring using only 2 colors. So $\sigma(FL_n) = 2$.

**Theorem 2.6.** The Friendship graph $F_n$ is $\sigma$-colorable and $\sigma(F_n) = 2$.

**Proof:** Let the central vertex of the Friendship graph $F_n$ be $v$ and let $v_1, v_2$ be the vertices of the first copy of $C_3$, $v_{21}, v_{22}$ be the vertices of the second copy of $C_3$, $v_{31}, v_{32}$ be the vertices of the third copy of $C_3$ and so on. Let $v_{n1}, v_{n2}$ be the vertices of the $n^{th}$ copy of $C_3$.

Define $C : V(F_n) \rightarrow \{1,2\}$ as follows.

$C(v) = 1$

$C(v_{2i}) = 1$ if $1 \leq i \leq n$.

$C(v_{2i-1}) = 2$ if $1 \leq i \leq n$.

Then this coloring is a minimal $\sigma$ – coloring using only 2 colors. So $\sigma(F_n) = 2$.

**Theorem 2.7.** The Double Wheel graph, $DW_n$ is $\sigma$-colorable and $\sigma(DW_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$

**Proof:** Let $v$ be the apex vertex. Let $\{v_1, v_2, v_3, ..., v_n\}$ and $\{u_1, u_2, u_3, ..., u_n\}$ be vertices of inner and outer cycles of $C_n$ respectively.

**Case 1.** $n > 3$ and $n$ is even

Let $v$ be the central vertex.

Define $C : V(DW_n) \rightarrow \{1,2\}$ as follows.

$C(v) = 1$

$C(v_{2i}) = 2$ if $1 \leq i \leq \frac{n}{2}$.

$C(v_{2i-1}) = 1$ if $1 \leq i \leq \frac{n}{2}$.

$C(u_{2i}) = 2$ if $1 \leq i \leq \frac{n}{2}$.

$C(u_{2i-1}) = 1$ if $1 \leq i \leq \frac{n}{2}$.

This coloring is a minimal $\sigma$ – coloring using only 2 colors. So $\sigma(DW_n) = 2$.

**Case 2.** $n$ is odd

Define $C : V(DW_n) \rightarrow \{1,2,3\}$ as follows.

$C(v) = 1$

$C(v_{3i}) = 2$ if $1 \leq i \leq \frac{n}{3}$.

$C(v_{3i-1}) = 1$ if $1 \leq i \leq \frac{n+1}{3}$.

$C(v_{3i-2}) = 3$ if $1 \leq i \leq \frac{n+2}{3}$.

$C(u_{3i}) = 2$ if $1 \leq i \leq \frac{n}{3}$.

$C(u_{3i-1}) = 1$ if $1 \leq i \leq \frac{n+1}{3}$.

$C(u_{3i-2}) = 3$ if $1 \leq i \leq \frac{n+2}{3}$.

This coloring is a minimal $\sigma$ – coloring using only 2 colors. So $\sigma(DW_n) = 3$.

**Theorem 2.8.** The Crown graph $C_n^+$ is $\sigma$-colorable and $\sigma(C_n^+) = 2$.

**Proof:** Let the vertices on the cycle be $v_1, v_2, v_3, ..., v_n$ and the pendent vertices corresponding to the cycle be $w_1, w_2, w_3, ..., w_n$.

**Case 1.** $n > 3$ and $n$ is even

Define $C : V(C_n^+) \rightarrow \{1,2\}$ as follows.

$C(v_{2i}) = 2$ if $1 \leq i \leq \frac{n}{2}$.

$C(v_{2i-1}) = 1$ if $1 \leq i \leq \frac{n}{2}$.

$C(w_i) = 1$ if $1 \leq i \leq n$.
This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(C_n) = 2$

**Case 2.** $n > 3$ and $n$ is odd
Define $C : V(C_n^+); \{1,2\}$ as follows.
$C(v_2i) = 2$ if $1 \leq i \leq \frac{n-1}{2}$.
$C(v_2i-1) = 1$ if $1 \leq i \leq \frac{n+1}{2}$.
$C(w_i) = 1$ if $1 \leq i \leq n - 1$.
$C(w_n) = 2$.

This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(C^+_n) = 2$

**Theorem 2.9.** The Double Crown graph, $C_n^+$, is $\sigma$-colorable and $\sigma(C_n^+) = 2$.

**Proof:** Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of the cycle $C_n$. Let $\{v_1, v_{12}\}$ be the pendent edge corresponding to each vertex, $v_i$, $1 \leq i \leq n$.

**Case 1.** $n > 3$ and $n$ is even
Define $C : V(C_n^+) \rightarrow \{1,2\}$ as follows.
$C(v_2i) = 2$ if $1 \leq i \leq \frac{n}{2}$ ; $C(v_2i-1) = 1$ if $1 \leq i \leq \frac{n+1}{2}$.
$C(v_{1i}) = 1 \Longleftrightarrow C(v_{12}) \quad$ if $1 \leq i \leq n$.

**Case 2.** $n > 3$ and $n$ is odd
Define $C : V(C_n^+) \rightarrow \{1,2\}$ as follows.
$C(v_2i) = 2$ if $1 \leq i \leq \frac{n-1}{2}$ ; $C(v_2i-1) = 1$ if $1 \leq i \leq \frac{n+1}{2}$.
$C(v_{1i}) = 1$ if $1 \leq i \leq n$ ; $C(v_{12}) = 1 \quad$ if $1 \leq i \leq n - 1$ ; $C(v_{n2}) = 2$.

This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(C_n^+) = 2$

**Theorem 2.10.** The Web graph, $W_n$ is $\sigma$-colorable and $\sigma(W_n) = 2$.

**Proof:** Let the central vertex of the Web graph $W_n$ be $v$. Let the vertices on the inner cycle be $v_1, v_2, v_3, \ldots, v_n$ and the vertices on the outer cycle be $u_1, u_2, u_3, \ldots, u_n$ and the pendent vertices be $w_1, w_2, w_3, \ldots, w_n$.

**Case 1.** $n > 3$ and $n$ is even
Define $C : V(W_n) \rightarrow \{1,2\}$ as follows:
$C(v) = 1$
$C(v_2i) = 2$ if $1 \leq i \leq \frac{n}{2}$ ; $C(v_2i-1) = 1$ if $1 \leq i \leq \frac{n}{2}$.
$C(v_{1i}) = 1$ if $1 \leq i \leq \frac{n}{2}$ ; $C(u_{2i}) = 2$ if $1 \leq i \leq \frac{n}{2}$.
$C(w_i) = 1$ if $1 \leq i \leq n$.

This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(W_n) = 2$

**Case 2.** $n = 5$
Define $C : V(W_n) \rightarrow \{1,2\}$ as follows:
$C(v) = 2$
$C(v_2) = 2, C(v_3) = 2, C(v_4) = 1, C(v_5) = 2$
$C(u_2) = 1, C(u_3) = 1, C(u_4) = 1, C(u_5) = 2$
$C(w_1) = 1$ if $1 \leq i \leq 3, C(w_4) = 2, C(w_5) = 1$

This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(W_n) = 2$

**Case 3.** $n > 5$ and $n$ is odd
Define $C : V(W_n) \rightarrow \{1,2\}$ as follows.
$C(v) = 1$
$C(v_2i) = 2$ if $1 \leq i \leq \frac{n-1}{2}$ ; $C(v_2i-1) = 1$ if $1 \leq i \leq \frac{n+1}{2}$.
$C(u_2i) = 1$, if $1 \leq i \leq \frac{n-1}{2}$ ; $C(u_2i-1) = 2$, if $1 \leq i \leq \frac{n-1}{2}$ ; $C(u_2i) = 1$.
$C(w_i) = 1$ if $1 \leq i \leq n - 1$ ; $C(w_n) = 2$.

This coloring is a minimal $\sigma$–coloring using only 2 colors. So $\sigma(W_n) = 2$

**References:**


