OSCILLATION OF THIRD ORDER HALF LINEAR DIFFERENCE EQUATION IN NEUTRAL TYPE

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Abstract: This paper discusses oscillatory behavior of the third order non-linear difference equations.

Key words –oscillation, third order, nonlinear, difference equation.

Introduction:

In this paper, we study the half linear neutral difference equations

$$\Delta\left[a(n)\left[\Delta^2\left[x(n)-p(n)x(\tau(n))\right]\right]^{\gamma}\right]+q(n)x^{\gamma}(\sigma(n))=0 \quad : n\geq n_0 \tag{1}$$

We define the function $z(n)=x(n)-p(n)x(\tau(n))$. (2)

Here we will assume that the following conditions are satisfied.

H_1. $a(n),p(n)\in C([n_0,\infty),R^{+}),q(n)\in C([n_0,\infty),R),\gamma>0$ is the quotient of odd positive integers.

H_2. $\tau(n),\sigma(n)$ are the continuous sequence $\sigma(n)<n,\lim_{n\to\infty}\tau(n)=\infty,\lim_{n\to\infty}\sigma(n)=\infty$

H_3. $\sum_{N}^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}=\infty$

where a(n) is a continuous positive sequence. By the solution of (1) we mean a nontrivial sequence $x(n)\in C([N_x,\infty),R),N_x\geq n_0$ for which $x(n)-p(n)x(\tau(n))\in C^2([N_x,\infty),R),a(n)(\Delta^2 z(n))^{\gamma}\in C^1([N_x,\infty),R),$ and (1) is satisfied on some interval $([N_x,\infty),R)$, where $N_x\geq n_0$. A non trivial solution of (1) is said to be oscillatory if it has arbitrarily large zeros, otherwise is said to be non oscillatory that is eventually positive solution or eventually negative solution. The purpose of this paper is to obtain necessary and sufficient conditions for thee oscillation of all solutions of (1).

Lemma:1

Suppose that $p,q\in C[R^{+},R^{+}],q(n)<n;n\geq n_0,\lim_{n\to\infty}q(n)=\infty$ and

$$\liminf_{n\to\infty}\sum_{q(n)}^{n-1}p(s)>\frac{1}{e} \tag{3}$$

Then the inequality $\Delta y(n)+p(n)y(q(n))\leq 0$ has no eventually positive solutions, and the inequality $\Delta y(n)+p(n)y(q(n))\geq 0$ has no eventually negative solutions.

Lemma:2

Suppose that H_1-H_3 holds, $q(n)\geq 0$ and let $x(n)$ be an eventually positive solution of (1) then there are only the following two cases for (2)

(i) $z(n)>0, \Delta z(n)>0, \Delta^2 z(n)>0, \left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]<0, n\geq n_1\geq n_0$

(ii) $z(n)>0, \Delta z(n)<0, \Delta^2 z(n)>0, \left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]<0, n\geq n_1\geq n_0$

Main Results:

In this section, we give the main results.

Theorem:1

Suppose that H_1-H_3 holds, $1\leq p(n)<p_1, \tau(n)>n, q(n)\leq 0$ and there exists continuous sequence , $\alpha(n), \beta(n)$ such that $\alpha(n)>n, \beta(n)>n$

$$\liminf_{n\to\infty}\sum_{s=F(n)}^{n-1}\sum_{v=s}^{\beta(s)-1}\left(\frac{1}{a(v)}\right)^{\frac{1}{\gamma}}\left[\sum_{w=v}^{\alpha(v)-1}\frac{|q(w)|}{p^{\gamma}(\tau^{-1}(\sigma(w)))}\right]^{\frac{1}{\gamma}}>\frac{1}{e} \tag{4}$$

$H(n)=\tau^{-1}(\alpha(\beta(n)))<n$

$$\liminf_{n\to\infty}\sum_{s=n}^{\alpha(n)-1}|q(s)|>0 \tag{5}$$

$$\sum_{s=n_1}^{\infty}\frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}=\infty, n\geq N \tag{6}$$

Then every solution of (1) is oscillatory.

Proof: Suppose that (1) has eventually positive solution $x(n)$ then we have

$\Delta\left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]\geq 0$, so by lemma there are only the following three cases for (2)

(i) $z(n)<0, \Delta z(n)>0, \Delta^2 z(n)<0, \Delta\left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]\geq 0, n\geq n_1\geq n_0$

(ii) $z(n)>0, \Delta z(n)>0, \Delta^2 z(n)<0, \Delta\left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]\geq 0, n\geq n_1\geq n_0$

(iii) $z(n)<0, \Delta z(n)<0, \Delta^2 z(n)<0, \Delta\left[a(n)\left(\Delta^2 z(n)\right)^{\gamma}\right]\geq 0, n\geq n_1\geq n_0$

Case (i): From equation (2) we follows that

$z(n)=x(n)-p(n)x(\tau(n))$

$x(n)=z(n)+p(n)x(\tau(n))$

$$x(\tau(n))=\frac{x(n)-z(n)}{p(n)}$$

$$x(\tau(n))=\frac{x(n)}{p(n)}-\frac{z(n)}{p(n)}$$

$$x(\tau(n)) > -\frac{z(n)}{p(n)}$$

$$x(n) > -\frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))}$$

$$x(\sigma(n)) > -\frac{z(\tau^{-1}(\sigma(n)))}{p(\tau^{-1}(\sigma(n)))} \tag{7}$$

Summing (1) from n to $\alpha(n)-1$, we get

$$\sum_{s=n}^{\alpha(n)-1} \Delta\left[a(s)\left[\Delta^2\left[x(s)-p(s)x(\tau(s))\right]\right]^{\gamma}\right] + \sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s)) = 0$$

$$\sum_{s=n}^{\alpha(n)-1} \Delta\left[a(s)\left[\Delta^2\left[x(s)-p(s)x(\tau(s))\right]\right]^{\gamma}\right] = -\sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s))$$

$$a(\alpha(n))\left[\Delta^2 z(\alpha(n))\right]^{\gamma} - a(n)\left[\Delta^2 z(n)\right]^{\gamma} = -\sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s))$$

By using equation (7), we get

$$-a(n)\left[\Delta^2 z(n)\right]^{\gamma} = -\sum_{s=n}^{\alpha(n)-1} |q(s)|\, x^{\gamma}(\sigma(s))$$

$$\geq -\sum_{s=n}^{\alpha(n)-1} |q(s)| \frac{z^{\gamma}(\tau^{-1}(\sigma(s)))}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

$$\geq -z^{\gamma}(\tau^{-1}(\sigma(\alpha(n)))) \sum_{s=n}^{\alpha(n)-1} \frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

$$a(n)\left[\Delta^2 z(n)\right]^{\gamma} \leq z^{\gamma}(\tau^{-1}(\sigma(\alpha(n)))) \sum_{s=n}^{\alpha(n)-1} \frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

$$\left[\Delta^2 z(n)\right]^{\gamma} \leq \frac{z^{\gamma}(\tau^{-1}(\sigma(\alpha(n))))}{a(n)} \sum_{s=n}^{\alpha(n)-1} \frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

$$\left[\Delta^2 z(n)\right] \leq \left(\frac{z^{\gamma}(\tau^{-1}(\sigma(\alpha(n))))}{a(n)}\right)^{\frac{1}{\gamma}} \left(\sum_{s=n}^{\alpha(n)-1} \frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}\right)^{\frac{1}{\gamma}}$$

$$\left[\Delta^2 z(n)\right] \leq \left(\frac{z(\tau^{-1}(\sigma(\alpha(n))))}{a^{\frac{1}{\gamma}}(n)}\right) \left(\sum_{s=n}^{\alpha(n)-1} \frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}\right)^{\frac{1}{\gamma}}$$

Summing the last inequality from n to $\beta(n)-1$

$$\sum_{s=n}^{\beta(n)-1} \left[\Delta^2 z(s)\right] \leq \sum_{s=n}^{\beta(n)-1} \frac{z(\tau^{-1}(\sigma(\alpha(s))))}{a^{\frac{1}{\gamma}}(s)} \left(\sum_{v=s}^{\alpha(s)-1} \frac{|q(v)|}{p^{\gamma}(\tau^{-1}(\sigma(v)))}\right)^{\frac{1}{\gamma}}$$

$$\Delta z(\beta(n)) - \Delta z(n) \leq z(\tau^{-1}(\sigma(\alpha(\beta(n))))) \sum_{s=n}^{\beta(n)-1} \frac{1}{a^{\frac{1}{\gamma}}(s)} \left(\sum_{v=s}^{\alpha(s)-1} \frac{|q(v)|}{p^{\gamma}(\tau^{-1}(\sigma(v)))}\right)^{\frac{1}{\gamma}}$$

$$-\Delta z(n) \le z(\tau^{-1}(\sigma(\alpha(\beta(n))))) \sum_{s=n}^{\beta(n)-1} \frac{1}{a^{\frac{1}{\gamma}}(s)} \left(\sum_{v=s}^{\alpha(s)-1} \frac{|q(v)|}{p^{\gamma}(\tau^{-1}(\sigma(v)))} \right)^{\frac{1}{\gamma}}$$

$$\Delta z(n) \ge -z(\tau^{-1}(\sigma(\alpha(\beta(n))))) \sum_{s=n}^{\beta(n)-1} \frac{1}{a^{\frac{1}{\gamma}}(s)} \left(\sum_{v=s}^{\alpha(s)-1} \frac{|q(v)|}{p^{\gamma}(\tau^{-1}(\sigma(v)))} \right)^{\frac{1}{\gamma}}$$

$$\Delta z(n) + z(H(n)) \sum_{s=n}^{\beta(n)-1} \frac{1}{a^{\frac{1}{\gamma}}(s)} \left(\sum_{v=s}^{\alpha(s)-1} \frac{|q(v)|}{p^{\gamma}(\tau^{-1}(\sigma(v)))} \right)^{\frac{1}{\gamma}} \ge 0$$

Where $H(n) = \tau^{-1}(\sigma(\alpha(\beta(n))))$

By lemma and condition (4) the last inequality cannot has eventually negative solution. Which is contradiction.

Case (ii) From (2) we get $x(n) > z(n)$, $n \ge n_1 \ge n_0$

$$x(\sigma(n)) > z(\sigma(n)) \tag{8}$$

Summing equation (1) from n to $\alpha(n)-1$

$$\sum_{s=n}^{\alpha(n)-1} \Delta\left[a(s)\left[\Delta^2\left[x(s) - p(s)x(\tau(s))\right]\right]^{\gamma} \right] + \sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s)) = 0$$

$$\sum_{s=n}^{\alpha(n)-1} \Delta\left[a(s)\left[\Delta^2\left[x(s) - p(s)x(\tau(s))\right]\right]^{\gamma} \right] = -\sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s))$$

$$a(\alpha(n))\left[\Delta^2 z(\alpha(n))\right]^{\gamma} - a(n)\left[\Delta^2 z(n)\right]^{\gamma} = -\sum_{s=n}^{\alpha(n)-1} q(s)x^{\gamma}(\sigma(s))$$

By using equation (8), we get

$$-a(n)\left[\Delta^2 z(n)\right]^{\gamma} \ge -\sum_{s=n}^{\alpha(n)-1} q(s)z^{\gamma}(\sigma(s))$$

$$a(n)\left[\Delta^2 z(n)\right]^{\gamma} \le \sum_{s=n}^{\alpha(n)-1} |q(s)|\, z^{\gamma}(\sigma(s))$$

$$\left[\Delta^2 z(n)\right]^{\gamma} \le \frac{1}{a(n)} \sum_{s=n}^{\alpha(n)-1} |q(s)|\, z^{\gamma}(\sigma(s))$$

$$\left[\Delta^2 z(n)\right] \le \frac{1}{a^{\frac{1}{\gamma}}(n)} \left(\sum_{s=n}^{\alpha(n)-1} |q(s)|\, z^{\gamma}(\sigma(s)) \right)^{\frac{1}{\gamma}}$$

$$\left[\Delta^2 z(n)\right] \le \frac{z(\sigma(n))}{a^{\frac{1}{\gamma}}(n)} \left(\sum_{s=n}^{\alpha(n)-1} |q(s)| \right)^{\frac{1}{\gamma}}$$

Summing the last inequality from n_1 to $n-1$

$$\sum_{s=n_1}^{n-1}\Delta^2 z(s) \le \sum_{s=n_1}^{n-1}\frac{z(\sigma(s))}{a^{\frac{1}{\gamma}}(s)}\left(\sum_{v=s}^{\alpha(s)-1}|q(v)|\right)^{\frac{1}{\gamma}}$$

$$z(n)-z(n_1) \le \sum_{s=n_1}^{n-1}\frac{z(\sigma(s))}{a^{\frac{1}{\gamma}}(s)}\left(\sum_{v=s}^{\alpha(s)-1}|q(v)|\right)^{\frac{1}{\gamma}}$$

$$z(n)-z(n_1) \le z(\sigma(n))\sum_{s=n_1}^{n-1}\frac{1}{a^{\frac{1}{\gamma}}(s)}\left(\sum_{v=s}^{\alpha(s)-1}|q(v)|\right)^{\frac{1}{\gamma}}$$

As $n \to \infty$ and in view of condition H_3 and equation (5) the last inequality leads to a contradiction.

Case (iii) In this case $a(n)\left(\Delta^2 z(n)\right)^{\gamma} < 0$ and non decreasing for $n \ge n_1$ hence it is bounded.

Summing equation (1) from n_1 to $n-1$

$$\sum_{s=n_1}^{n-1}\Delta\left[a(s)\left[\Delta^2\left[x(s)-p(s)x(\tau(s))\right]\right]^{\gamma}\right]+\sum_{s=n_1}^{n-1}q(s)x^{\gamma}(\sigma(s))=0$$

$$\sum_{s=n_1}^{n-1}\Delta\left[a(s)\left[\Delta^2\left[z(n)\right]\right]^{\gamma}\right]=-\sum_{s=n_1}^{n-1}q(s)x^{\gamma}(\sigma(s))$$

$$a(n)\left[\Delta^2 z(n)\right]^{\gamma}-a(n_1)\left[\Delta^2 z(n_1)\right]^{\gamma}=-\sum_{s=n_1}^{n-1}q(s)x^{\gamma}(\sigma(s))$$

By using equation (7), we get

$$-a(n_1)\left[\Delta^2 z(n_1)\right]^{\gamma}=-\sum_{s=n_1}^{n-1}|q(s)|\,x^{\gamma}(\sigma(s))$$

$$\ge -\sum_{s=n_1}^{n-1}|q(s)|\frac{z^{\gamma}(\tau^{-1}(\sigma(s)))}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

$$\ge -z^{\gamma}(\tau^{-1}(\sigma(n)))\sum_{s=n_1}^{n-1}\frac{|q(s)|}{p^{\gamma}(\tau^{-1}(\sigma(s)))}$$

As $n \to \infty$ and in view of (6) the last inequality leads to contradiction.

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