Asymptotes Concavity and Inflection Points, Second Derivative Test - An Analysis

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Abstract

This paper attempts to study how curvature rigorously, function f differentiable on (a, b) is called X concave up (or convex) if f ′ is increasing on (a, b); X concave down (or concave) if f ′ is decreasing on (a, b). Sign of the derivative tells us whether a function is increasing or decreasing; for example, when f′(x)>0f′(x)>0, f(x)f(x) is increasing. The sign of the second derivative f′′(x)f′′(x) tells us whether f′f′ is increasing or decreasing; we have seen that if f′f′ is zero and increasing at a point then there is a local minimum at the point, and if f′f′ is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about ff from information about f′′f′′.

To get information from the sign of f′′f′′ even when f′f′ is not zero. Suppose that f′′(a)>0f′′(a)>0. This means that near x=ax=a, f′f′ is increasing. If f′(a)>0f′(a)>0, this means that ff slopes up and is getting steeper; if f′(a)<0f′(a)<0, this means that ff slopes down and is getting *less* steep. A curve that is shaped like this is called concave up. knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called inflection points. If the concavity changes from up to down at x=ax=a, f′′f′′ changes from positive to the left of aa to negative to the right of aa, and usually f′′(a)=0f′′(a)=0. To identify such points by first finding where f′′(x)f′′(x) is zero and then checking to see whether f′′(x)f′′(x) does in fact go from positive to negative or negative to positive at these points. Note that it is possible that f′′(a)=0f′′(a)=0 but the concavity is the same on both sides. The intervals of concavity can be found in the same way used to determine the intervals of increase/decrease, except that we use the second derivative instead of the first. In particular, since $(f')'=f"$, the intervals of increase/decrease for the first derivative will determine the concavity of f. Take the number line showing subcritical numbers and intervals of concavity from the process above. The points (s,f(s)) where the concavity changes are inflection points. Thus not all subcritical numbers will yield inflection points (just like not all critical numbers yield local extrema).

Key words: Asymptotes, function , Concavity, differentiable, inflection points, second derivative test

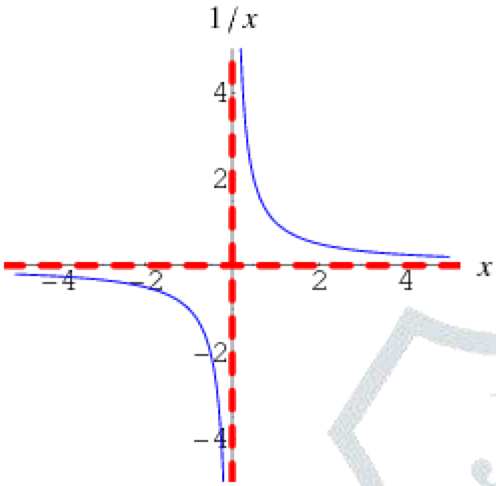
Introduction

A point of inflection of the graph of a function ff is a point where the *second* derivative f′′f′′ is 00. We have to wait a minute to clarify the geometric meaning of this.

A piece of the graph of ff is concave upward if the curve ‘bends’ upward. For example, the popular parabola y=x2y=x2 is concave upward in its entirety.

A piece of the graph of ff is concave downward if the curve 'bends' downward. For example, a 'flipped' version y=−x2y=−x2 of the popular parabola is concave downward in its entirety.

An asymptote is a line or curve that approaches a given curve arbitrarily closely, as illustrated in the above diagram.



The plot above shows $1/x$, which has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

The relation of *points of inflection* to *intervals where the curve is concave up or down* is exactly the same as the relation of *critical points* to *intervals where the function is increasing or decreasing*. That is, the points of inflection mark the boundaries of the two different sort of behavior. Further, only one sample value of f″f″ need be taken between each pair of consecutive inflection points in order to see whether the curve bends up or down along that interval.

Expressing this as a systematic procedure: *to find the intervals along which* ff *is concave upward and concave downward:*

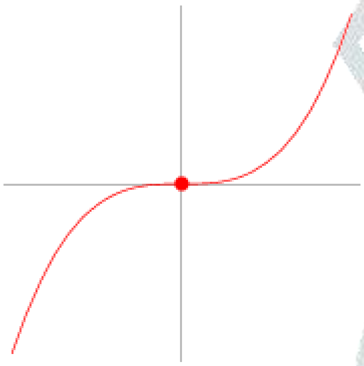
- Compute the *second* derivative f″f″ of ff, and *solve* the equation f″(x)=0f″(x)=0 for xx to find all the inflection points, which we list in order as x1<x2<…<xnx1<x2<…<xn. (Any points of discontinuity, etc., should be added to the list!)
- We need some *auxiliary points*: To the left of the leftmost inflection point x1x1 pick any convenient point toto, between each pair of consecutive inflection points xi,xi+1xi,xi+1 choose any convenient point titi, and to the right of the rightmost inflection point xnxn choose a convenient point tntn.
- Evaluate the *second derivative* f″f″ at all the *auxiliary* points titi.
- Conclusion: if f″(ti+1)>0f″(ti+1)>0, then ff is *concave upward* on (xi,xi+1)(xi,xi+1), while if f″(ti+1)<0f″(ti+1)<0, then ff is *concave downward* on that interval.

- Conclusion: on the 'outside' interval (−∞,xo)(−∞,xo), the function ff is *concave upward* if f″(to)>0f″(to)>0 and is *concave downward* if f″(to)<0f″(to)<0. Similarly, on (xn,∞)(xn,∞), the function ff is *concave upward* if f″(tn)>0f″(tn)>0 and is *concave downward* if f″(tn)<0f″(tn)<0.

Objective:

This paper intends to explore and analyze an inflection point, a point on a smooth plane curve at which the curvature changes sign. In particular, in the case of the graph of a function, it is a point where the function changes from being concave (concave downward) to convex (concave upward), or vice versa.

inflection point



An inflection point is a point on a curve at which the sign of the curvature (i.e., the concavity) changes. Inflection points may be stationary points, but are not local maxima or local minima. For example, for the curve $y = x^3$ plotted above, the point $x = 0$ is an inflection point.

The first derivative test can sometimes distinguish inflection points from extrema for differentiable functions $f(x)$.

The second derivative test is also useful. A necessary condition for x to be an inflection point is $f''(x) = 0$. A sufficient condition requires $f''(x+\epsilon)$ and $f''(x-\epsilon)$ to have opposite signs in the neighborhood of x (Bronshtein and Semendyayev 2004, p. 231).

For the graph of a function of differentiability class C^2 (*f*, its first derivative *f*′, and its second derivative *f*″, exist and are continuous), the condition $f'' = 0$ can also be used to find an inflection point since a point of $f'' = 0$ must be passed to change *f*″ from a positive value (concave upward) to a negative value (concave downward) or vise versa as *f*″ is continuous; an inflection point of the curve is where $f'' = 0$ and changes its sign at the point (from positive to negative or from negative to positive).[1] A point where the second derivative vanishes but does not change its sign is sometimes called a point of undulation or undulation point.

In algebraic geometry an inflection point is defined slightly more generally, as a regular point where the tangent meets the curve to order at least 3, and an undulation point or hyperflex is defined as a point where the tangent meets the curve to order at least 4.

For a function *f*, if its second derivative $f''(x)$ exists at x_0 and x_0 is an inflection point for *f*, then $f''(x_0) = 0$, but this condition is not sufficient for having a point of inflection, even if derivatives of any order exist. In this case, one also needs the lowest-order (above the second) non-zero derivative to be of odd order (third, fifth, etc.). If the lowest-order non-zero derivative is of even order, the point is not a point of inflection, but an *undulation point*. However, in algebraic geometry, both inflection points and undulation points are usually called *inflection points*. An example of an undulation point is $x = 0$ for the function *f* given by $f(x) = x^4$.

In the preceding assertions, it is assumed that *f* has some higher-order non-zero derivative at *x*, which is not necessarily the case. If it is the case, the condition that the first nonzero derivative has an odd order implies that the sign of $f'(x)$ is the same on either side of *x* in a neighborhood of *x*. If this sign is positive, the point is a *rising point of inflection*; if it is negative, the point is a *falling point of inflection*.

Inflection points sufficient conditions:

1) A sufficient existence condition for a point of inflection in the case that $f(x)$ is *k* times continuously differentiable in a certain neighborhood of a point x_0 with *k* odd and $k \geq 3$, is that $f^{(n)}(x_0) = 0$ for $n = 2, \ldots, k-1$ and $f^{(k)}(x_0) \neq 0$. Then $f(x)$ has a point of inflection at x_0.

2) Another more general sufficient existence condition requires $f''(x_0 + \varepsilon)$ and $f''(x_0 - \varepsilon)$ to have opposite signs in the neighborhood of x_0 (Bronshtein and Semendyayev 2004, p. 231).

Points of inflection can also be categorized according to whether $f'(x)$ is zero or nonzero.

- if $f'(x)$ is zero, the point is a *stationary point of inflection*
- if $f'(x)$ is not zero, the point is a *non-stationary point of inflection*

A stationary point of inflection is not a local extremum. More generally, in the context of functions of several real variables, a stationary point that is not a local extremum is called a saddle point.

An example of a stationary point of inflection is the point (0, 0) on the graph of $y = x^3$. The tangent is the *x*-axis, which cuts the graph at this point.

An example of a non-stationary point of inflection is the point (0, 0) on the graph of $y = x^3 + ax$, for any nonzero *a*. The tangent at the origin is the line $y = ax$, which cuts the graph at this point

Some functions change concavity without having points of inflection. Instead, they can change concavity around vertical asymptotes or discontinuities. For example, the function is concave for negative *x* and convex for positive *x*, but it has no points of inflection because 0 is not in the domain of the function.

Functions with inflection points whose second derivative does not vanish

Some continuous functions have an inflection point even though the second derivative is never 0. For example, the cube root function is concave upward when x is negative, and concave downward when x is positive, but has no derivatives of any order at the origin.

Suppose $f(x)$ is a function of x that is twice differentiable at a stationary point x_0.

1. If $f''(x_0) > 0$, then f has a local minimum at x_0.

2. If $f''(x_0) < 0$, then f has a local maximum at x_0.

The extremum test gives slightly more general conditions under which a function with $f''(x_0) = 0$ is a maximum or minimum.

If $f(x, y)$ is a two-dimensional function that has a local extremum at a point (x_0, y_0) and has continuous partial derivatives at this point, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. The second partial derivatives test classifies the point as a local maximum or local minimum.

Define the second derivative test discriminant as

$$D \equiv f_{xx} f_{yy} - f_{xy} f_{yx} \quad (1)$$

$$= f_{xx} f_{yy} - f_{xy}^2. \quad (2)$$

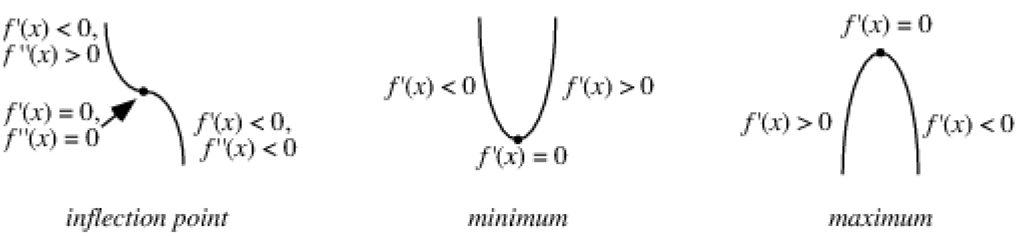
Then

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, the point is a local minimum.

2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, the point is a local maximum.

3. If $D < 0$, the point is a saddle point.

4. If $D = 0$, higher order tests must be used



Suppose $f(x)$ is continuous at a stationary point x_0.

1. If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then $f(x)$ has a local maximum (possibly a global maximum) at x_0.

2. If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then $f(x)$ has a local minimum (possibly a global minimum) at x_0.

3. If $f'(x)$ has the same sign on an open interval extending left from x_0 and on an open interval extending right from x_0, then $f(x)$ has an inflection point at x_0.

the normal curvature is 0 in the direction $\mathbf{x}'(t)$ for all t in the domain of $\mathbf{x}$. The differential equation for the parametric representation of an asymptotic curve is

$$e\,u'^2 + 2\,f\,u'\,v' + g\,v'^2 = 0, \tag{1}$$

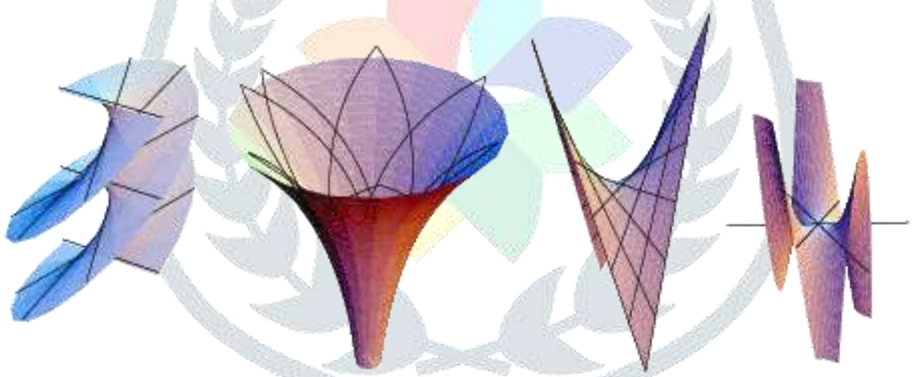
where e, f, and g are coefficients of the second fundamental form. The differential equation for asymptotic curves on a Monge patch $(u, v, h(u, v))$ is

$$h_{uu}\,u'^2 + 2\,h_{uu}\,u'\,v' + h_{vv}\,v'^2 = 0, \tag{2}$$

and on a polar patch $(r\cos\theta, r\sin\theta, h(r))$ is

$$h''(r)\,r'^2 + h'(r)\,r\,\theta'^2 = 0. \tag{3}$$

The images below show asymptotic curves for the elliptic helicoid, funnel, hyperbolic paraboloid, and monkey saddle.



Conclusion

derivative test uses the derivatives of a function to locate the critical points of a function and determine whether each point is a local maximum, a local minimum, or a saddle point. Derivative tests can also give information about the concavity of a function.The first-derivative test examines a function's monotonic properties (where the function is increasing or decreasing), focusing on a particular point in its domain. If the function "switches" from increasing to decreasing at the point, then the function will achieve a highest value at that point. Similarly, if the function "switches" from decreasing to increasing at the point, then it will achieve a least value at that point. If the function fails to "switch" and remains increasing or remains decreasing, then no highest or least value is achieved. One can examine a function's monotonicity without calculus. However, calculus is usually helpful because there are sufficient conditions that guarantee the monotonicity properties above, and these conditions apply to the vast majority of functions one would encounter. Stated precisely, suppose that f is a continuous real-valued function of a real variable, defined on some open interval containing the point x.

- If there exists a positive number r > 0 such that f is weakly increasing on (x − r, x] and weakly decreasing on [x, x + r), then f has a local maximum at x. This statement also works the other way around, if x is a local maximum point, then f is weakly increasing on (x − r, x] and weakly decreasing on [x, x + r).
- If there exists a positive number r > 0 such that f is strictly increasing on (x − r, x] and strictly increasing on [x, x + r), then f is strictly increasing on (x − r, x + r) and does not have a local maximum or minimum at x.

This statement is a direct consequence of how local extrema are defined. That is, if x0 is a local maximum point, then there exists r > 0 such that f(x) ≤ f(x0) for x in (x0 − r, x0 + r), which means that f has to increase from x0 − r to x0 and has to decrease from x0 to x0 + r because f is continuous. In the first two cases, f is not required to be strictly increasing or strictly decreasing to the left or right of x, while in the last two cases, f is required to be strictly increasing or strictly decreasing. The reason is that in the definition of local maximum and minimum, the inequality is not required to be strict: e.g. every value of a constant function is considered both a local maximum and a local minimum.

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