SUBMANIFOLDS IMMERSED IN A GENERALISED QUATERNION MANIFOLD

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Abstract:
Quaternion submanifolds of codimension 2 have been defined and studied by Hamoui[4] which is extension to the study done by other researchers. Almost r-contact structure was studied by Vanzura[12] and Yano and Ako[13]. The aim of the present paper is to study the submanifolds of codimension r immersed in a generalised quaternion manifold. Some interesting results have been stated and proved in this paper.

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1. Introduction

Yano and Ishihara [14] studied invariant submanifolds of an almost contact manifold. Yano and Ako[13] and Vanzura[12] further studied integrability conditions of an almost quaternion structure. Dube and Nivas[3] extended this study to almost r-contact hyperbolic structure in a product manifold. Hamouni[4] studied quaternion submanifold of codimension 2. Mishra[7] studied different structures on differentiable manifolds. Nivas with other researchers **Error! Reference source not found.**,[10],[1],[2] studied submanifolds of codimension 2 immersed in Hsu quaternion manifold. Salamon[11] developed the idea of quaternionic manifold, Pantilie[9] made a detailed study of generalised quaternionic manifold which was extended further by Deschamps. Jafri and other researchers [5],[6] studied algebraic properties of generalized quaternions. In this research we have studied the submanifolds of codimension r immersed in a generalised quaternion manifold. Some interesting results on submanifolds immersed in a generalised quaternion manifold have been stated and proved in this research.
Let M^{4n} be 4n-dimensional differentiable manifold of class C^∞. Let$\{F^*, G^*, H^*\}$ be a set of (1,1) tensor fields on M^{4n} satisfying the following relations from [1]

$$F^{*2} = a^2 I, G^{*2} = a^2 I, H^{*2} = a^2 I \tag{1.1}$$

I denotes the unit tensor field and a is non zero complex number. Also

$$\begin{aligned} (i) aF^* &= G^* H^* = -H^* G^* \\ (ii) aG^* &= H^* F^* = -F^* H^* \\ (iii) aH^* &= F^* G^* = -G^* F^* \end{aligned} \tag{1.2}$$

Let us call such a manifold M^{4n} satisfying (1.1) and (1.2) as the generalised quanterion manifold. Let g be the Riemannian metric on M^{4n} with respect to the tensor field F^*. Then obiviously

$$g(F^* X^*, F^* Y^*) + a^2 (g)(X^*, Y^*) = 0 \tag{1.3}$$

for arbitrary vector fields X^*, Y^* on M^{4n}.
A manifold V_n is said to possess a generalised para r-contact structure from[7] if there exists a tensor field f of type $(1,1), r(C^\infty)$ contravariant vector fields $\underset{x}{U}$ and $r(C^\infty)$ 1-forms $\overset{x}{u}$ (r some finite integer) satisfying

$$f^2 = a^2 I - \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \tag{1.4}$$

Also

$$\begin{aligned} (i)\ \overset{y}{u} o f + \sum_{x=1}^{r} \theta_x^y \overset{x}{u} &= 0 \\ (ii)\ f \underset{x}{U} + \sum_{y=1}^{r} \theta_x^y \underset{y}{U} &= 0 \\ (iii)\ \overset{x}{u} \underset{y}{U} + \sum_{z=1}^{r} \theta_z^x \theta_y^z &= a^2 \delta_y^x \end{aligned} \tag{1.5}$$

where x, y, z = 1, 2,..., r, δ_y^x the Kronecker delta and θ_x^y are scalar fields.

2. Structures in the Submanifold M^{4n-r}

Let M^{4n-r} be a submanifold of codimension r of the generalised quaternion manifold M^{4n}. Let τ denotes the immersion $M^{4n-r} \to M^{4n}$ and $B = d\tau$. Thus a vector field X in the tangent space of M^{4n-r} corresponds to a vector field BX in to M^{4n} from . Let $\underset{x}{N}$, x = 1, 2,..., r be r mutually orthogonal unit normal to M^{4n-r}. The transformations F*BX and $F^* \underset{x}{N}$ can be in the form from **Error! Reference source not found.**

$$F^* BX = BFX - \sum_{x=1}^{r} \overset{x}{u}(X) \underset{x}{N} \tag{2.1}$$

where F is a tensor field of type (1,1) and $\overset{x}{u}$ are $r(C^\infty)$ 1- forms on the submanifold M^{4n-r}. Also

$$F^* \underset{x}{N} = -B \underset{x}{U} + \sum_{y=1}^{r} \theta_x^y \underset{y}{N} \tag{2.2}$$

$\underset{x}{U}$, x=1, 2,.., r are $r(C^\infty)$ vector fields on the generalised quaternion submanifold M^{4n-r}. Similarly for the tensor fields G^* and H^* we have the following sets of tranformations :

$$(i) G^* BX = BGX - \sum_{x=1}^{r} \overset{x}{v}(X) \underset{x}{N}$$

and

$$(ii)\ G^* \underset{x}{N} = -B \underset{x}{V} + \sum_{y=1}^{r} \theta_x^y \underset{y}{N} \tag{2.3}$$

$\overset{x}{v}$ and $\underset{x}{V}$ are $r(C^\infty)$ 1- forms and vector fields(x=1, 2, .., r) and G is the tensor field of type (1,1) on the submanifold M^{4n-r}. Similarly,

$$(i)\ H^* BX = BHX - \sum_{x=1}^{r} \overset{x}{w}(X) \underset{x}{N}$$

and

$$(ii)\ H^* \underset{x}{N} = -B \underset{x}{W} + \sum_{y=1}^{r} \theta_x^y \underset{y}{N} \tag{2.4}$$

$\overset{x}{w}$ and $\underset{x}{W}$ are $r(C^{\infty)}$ 1- forms and vector fields (x=1, 2, .., r) respectively and H is the tensor field of type (1,1) on the submanifold M^{4n-r}.

Theorem (2.1) The submanifold M^{4n-r} of codimension r of the generalised quaternion manifold M^{4n} admits the generalised para - r contact structure with respect to (1, 1) tensor field F.

Proof : Operating the equation (2.1) by F^* and making use of the equations $(1.1), (2.1)$ and (2.2) we get

$$a^2Bx = BF^2x - \sum_{y=1}^{r} \overset{y}{u}\,(FX)\underset{y}{N} - \sum_{x=1}^{r} \overset{x}{u}\,(X)\{-B\underset{x}{U} + \sum_{y=1}^{r} \theta_x^y\,\underset{y}{N}\}$$

Comparision of tangential and normal vectors yields

$$F^2 = a^2I - \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \tag{2.5}$$

and

$$\overset{y}{u}of + \sum_{x=1}^{r} \theta_x^y\,\overset{x}{u} = 0 \tag{2.6}$$

Multiplying the equation (2.2) by c and making use of the equations $(1.1), (2.1)$ and (2.2) we obtain

$$a^2\underset{x}{N} = -\{BF\underset{x}{U} - \sum_{z=1}^{r} \overset{z}{u}\left(\underset{x}{U}\right)\underset{z}{N}\} + \sum_{y=1}^{r} \theta_x^y\,\{-B\underset{y}{U} + \sum_{z=1}^{r} \theta_y^z\,\underset{z}{N}\}$$

Equating tangential and normal vectors we get

$$F\underset{x}{U} + \sum_{y=1}^{r} \theta_x^y\,\underset{y}{U} = 0 \tag{2.7}$$

and

$$\overset{z}{u}(\underset{x}{U}) + \sum_{y=1}^{r} \theta_y^z\,\theta_x^y = a^2\delta_x^z \tag{2.8}$$

In view of the equations $(2.5), (2.6)$, (2.7)and (2.8) it follows that the submanifold 'M^{4n-r}' admits the generalised para- r contact structure.

Theorem (2.2) The submanifold M^{4n-r} of the generalised quaternion manifold M^{4n} also admits the generalised para- r contact structure with respect to the (1,1) tensor fields G^* and H^*.

Proof : This follows in a way similar to that of the theorem (2.1).

3. Some other results

We have in view of the equation (1.2) (i)

$$F^*BX = G^*H^*BX \tag{3.1}$$

By virtue of the equations (2.1), (2.3) and (2.4) the above equation (3.1) takes the form

$$BFX - \sum_{x=1}^{r} \overset{y}{u}\,(X)\underset{y}{N} = BGHX - \sum_{y=1}^{r} \overset{y}{v}\,(HX)\underset{y}{N} - \sum_{x=1}^{r} \overset{x}{w}\,(X)\{-B\underset{x}{V} + \sum_{y=1}^{r} \theta_x^y\,\underset{y}{N}\}$$

Comparision of tangential and normal vectors yields

$$(i)\ GHX = FX - \sum_{x=1}^{r} \overset{x}{w}(X)\underset{x}{V}$$
and
$$(ii)\ \overset{y}{v}(HX) + \sum_{x=1}^{r} \theta_x^y\, \overset{x}{w}(X) = \overset{y}{u}(X) \tag{3.2}$$

where x, y, z = 1, 2,..., r.
We also have from the same equation (1.2)(i)

$$F^*\underset{x}{N} = G^*H^*\underset{x}{N}$$

The above equation in view of the equation (2.2), (2.3) and (2.4) takes the form

$$-B\underset{x}{U} + \sum_{z=1}^{r} \theta_x^z\, \underset{z}{N} = -BG\underset{x}{W} + \sum_{z=1}^{r} \overset{z}{v}(\underset{x}{W})\underset{z}{N} + \sum_{y=1}^{r} \theta_x^y \{-B\underset{y}{V} + \sum_{z=1}^{r} \theta_y^z - B\underset{z}{N}\}$$

equating tangential and normal fields both sides, we get

$$G\underset{x}{W} + \sum_{y=1}^{r} \theta_x^y\, \underset{y}{V} = \underset{x}{U} \tag{3.3}$$

and

$$\overset{z}{v}(\underset{x}{W}) + \sum_{y=1}^{r} \theta_x^y\, \theta_y^z = \theta_x^z \tag{3.4}$$

Similarly we can obtain

$$HF = G - \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{W} \tag{3.5}$$

$$FG = H - \sum_{x=1}^{r} \overset{x}{v} \otimes \underset{x}{U}, etc. \tag{3.6}$$

Further, in view of the relation (1.2) (i), it follows that

$$(G^*H^* + H^*G^*)BX = 0 \tag{3.7}$$

By virtue of the equations (2.3) and (2.4) the above equation (3.7) takes the form

$$BGHX - \sum_{y=1}^{r} \overset{y}{v}(HX)\underset{y}{N} - \sum_{x=1}^{r} \overset{x}{w}(X)\{-B\underset{x}{V} + \sum_{y=1}^{r} \theta_x^y\, \underset{y}{N}\} + BHGX - \sum_{y=1}^{r} \overset{y}{w}(GX)\underset{y}{N} - \sum_{x=1}^{r} \overset{x}{v}(X)\{-B\underset{x}{W} + \sum_{y=1}^{r} \theta_x^y\, \underset{y}{N}\} = 0$$

Comparision of tangential and normal vectors yields

$$(GH + HG)X + \sum_{x=1}^{r}\{\overset{x}{v}(X)\underset{x}{W} + \overset{x}{w}(X)\underset{x}{V}\} = 0 \tag{3.8}$$

and

$$\overset{y}{v}(HX) + \overset{y}{w}(GX) + \sum_{x=1}^{r} \theta_x^y\, \{\overset{x}{v}(X) + \overset{x}{w}(X)\} = 0 \tag{3.9}$$

Further we also have

$$(G^*H^* + H^*G^*)\underset{x}{N} = 0, x = 1,2,3,\ldots,r \tag{3.10}$$

In view of the same equations (2.3) and (2.4), the above equation (3.10) takes the form

$$-BG\underset{x}{W} + \sum_{z=1}^{r} \overset{z}{v}(\underset{x}{W})\underset{z}{N} + \sum_{y=1}^{r} \theta_x^y \{-B\underset{y}{V} - \sum_{z=1}^{r} \theta_y^z \underset{z}{N}\} - BH\underset{x}{V} +$$
$$\sum_{z=1}^{r} \overset{z}{w}(\underset{x}{V})\underset{z}{N} + \sum_{y=1}^{r} \theta_x^y \{-B\underset{y}{W} + \sum_{z=1}^{r} \theta_y^z \underset{z}{N}\} = 0$$

Comparision of tangential and normal vector fields, yields

$$G\underset{x}{W} + H\underset{x}{V} + \sum_{y=1}^{r} \theta_x^y \{\underset{y}{V} + \underset{y}{W}\} = 0 \qquad (3.11)$$

and

$$\overset{z}{v}(\underset{x}{W}) + \overset{z}{W}(\underset{x}{V}) + \sum_{z=1}^{r} 2\,\theta_x^y \theta_y^z = 0 \qquad (3.12)$$

For $x, y, z = 1,2,3, \dots, r$.
Similar results for other tensor fields can also be established in a similar manner.

4. Conclusion

In this research we have defined ganeralised quaternion manifold which posses generalised para r-contact structure. Further we have defined some structures and transformations in the submanifold M^{4n-r}. We have proved necessary conditions for the submanifold M^{4n-r} of codimension r of the generalised quaternion manifold M^{4n} admiting the generalised para - r contact structure with respect to (1, 1) tensor field F^*, G^* and H^*. Similar results are also proved for other defined structure.

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