Study of C-Set in a linear space

Dr. Kumar Swetank

Jai Prakash University, Chapra(Saran), Bihar, India

Abstract

I use the notion of C-Set in a linear space which is a generalization of a convex set. It is branch of functional analysis which deals with some advancements in the theory of linear space and allied topics in functional analysis.

Introduction

In this paper we have defined C-Set in a linear space which is a generalization of a convex set. It contains 0 element which is not true in case of a convex set. We have established some and several theorems regarding C-Sets

Let A be a non-empty subset of a Linear space L. If for $x, y \in A, \alpha x - \beta y \in A,$ where $0 \le \alpha$ and $0 \le \beta$ and $\alpha + \beta = 1$, Then we define A to be a C-Set.

Choosing $\alpha = \beta = \frac{1}{2}$, $x = y$ then $\frac{1}{2}x - \frac{1}{2}x = 0 \in A$ .Choosing $\alpha = 0, \beta = 1,$ we see that $-y \in A$. Also, taking –y in place of y, $\alpha x - \beta(-y) = \alpha x + \beta y \in A$. Hence a C-Set is also a Convex set.

But as it is clear from the definition, a convex set is not necessarily a C-Set. Thus a C-Set is more general than a convex set Also 0 is an element of every C-Set which is not true in case of a convex set. It is also clear that a non-empty set A is a C-Set if and only if for $\alpha \ge 0, \beta \ge 0, \alpha + \beta = 1, \alpha x \pm \beta y \in A$.

Now, we study some properties of C-Sets.

Theorem (I) : Intersection of a family of C-Sets in a linear space L is also a C-Set.

Proof : Let $\{A_i\}$ be a family of C-Sets in L. We will show that $\bigcap_i A_i$ is also a C-Set.

Let $x, y \in \bigcap_i A_i$ Then for each $i, x \in A_i$ and $y \in A_i$ Let $0 \le \alpha$, $0 \le \beta$ and $\alpha + \beta = 1$ Then $\alpha x - \beta y \in A_i$ for each A_i

Hence $\alpha x - \beta y \in \bigcap_i A_i$ Therefore $\bigcap_i A_i$ is a C-Set.

Theorem (II) : Let S_1, S_2 be a C-Sets in a linear space L. If k_1, k_2 are scalars, then $k_1S_1 + k_2S_2$ is also a C-Set.

Proof : Let z_1, z_2 be elements of $k_1S_1 + k_2S_2$.

Then we can write

$z_1 = k_1x_1 + k_2y_1$ for some $x_1 \in S_1, y_1 \in S_2$

$z_2 = k_1x_2 + k_2y_2$ for some $x_2 \in S_1, y_2 \in S_2$

Let $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Then

$$\alpha z_1 - \beta z_2 = \alpha(k_1x_1 + k_2y_1) - \beta(k_1x_2 + k_2y_2)$$

$$= k_1(\alpha x_1 - \beta x_2) + k_2(\alpha y_1 - \beta y_2)$$

Since $\alpha x_1 - \beta x_2 \in S_1$, and $\alpha y_1 - \beta y_2 \in S_2$

(S_1, S_2 being C-Sets), $\alpha z_1 - \beta z_2$ is an elements of $k_1S_1 + k_2S_2$

Hence $k_1S_1 + k_2S_2$ is also a C-Set.

COROLLARY: If L is a linear space and A is a C-Set in L then A–A is also a C–Set in L.

Theorem (III) : Let $\{A_i\}$ be a family of C–Sets such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \ldots\ldots$, that is, $\{A_i\}$ is totally ordered by set inclusion, then $\bigcup_i A_i$ is also a C-Set.

Proof : Let the partial order relation '$\leq$' be symbolized by

$$A_i \leq A_j \Leftrightarrow A_i \subseteq A_j.$$

Let $x, y \in \bigcup_i A_i$, then there exists i, j such that $x \in A_i, y \in A_j$. Since $\{A_i\}$ is totally ordered, We can assume without loss of generality that $A_i \leq A_j$.

Hence $A_i \subseteq A_j$

Thus $x, y \in A_j$

Let $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$

Since A_j is a C-Set.

$\alpha x - \beta y \in A_j$

$$\Rightarrow \alpha x - \beta y \in \bigcup_i A_i$$

Therefore $\bigcup_i A_i$ is a C-Set.

Theorem (IV) : The family P $= \{A_i\}$ of all C-Sets of a linear space L which are totally ordered by set inclusion, is a complete lattice.

Proof : Let us define the partial ordering '$\leq$' on $\{A_i\}$ by

$$A_i \leq A_j \Leftrightarrow A_i \subseteq A_j$$

Then clearly P is a partially ordered set. Also P is a non-empty, for the set {0} is a C-Set and is contained in any C-Set and hence is in P Now, if A_i & A_j are any two elements of P, then $A_i \bigcup A_j$ is the least upper bound and $A_i \bigcap A_j$ is the greatest lower bound of the family $\{A_i, A_j\}$.

Also $A_i \bigcap A_j \in$ P

and $A_i \bigcup A_j \in$ P

Thus P is a lattice.

Further let Q be any non-empty subfamily of P.

Let Q $= \{A_{i_\alpha}\}_\alpha$. Then clearly $\bigcap_\alpha A_{i_\alpha}$ is the greatest lower bound of Q and $\bigcap_\alpha A_{i_\alpha} \in$ P .Also $\bigcup_\alpha A_{i_\alpha}$ is the least upper bound and $\bigcup_\alpha A_{i_\alpha} \in$ P

Hence P is a complete lattice.

Theorem (V) : Every linear space has a maximal C-Set.

Proof : Let P $= \{A_i\}$ be the family of all C-Sets of a linear space L. Then $\{0\} \in$ P. It is also clear that P is a partially ordered set with set inclusion as the partial ordering.

Let Q be any totally ordered subset of P such that

Q $= \{A_{i_\alpha}\}$. Then we have seen in the proof of theorem (IV) that $\bigcup_\alpha A_{i_\alpha}$ is the least upper bound of Q and $\bigcup_\alpha A_{i_\alpha} \in$ P.

Thus every totally ordered subset of P is bounded above. Hence by Zorn's Lemma P has a maximal element.

Thus L has a maximal C-Set.

REFERENCES

1. Norman L. Levine : Semi-open sets and semi-continuity in topological spaces, American Math. Monthly 70 (1963), P.P. 36-41.

2. S. Gene crossley and S.K. Hildebrand, semi-closure, Texas J. Sci. 22 (2+3)(1971), P.P. 99-112.

3. O. NJASTAD, on some classes of nearly open sets, pacific J. Math. 15 (1965), 861-870.

4. Dunford, N. and Schwartz, J. : "Linear operators Part-I", Interscience Publishers, Inc., New York, 1967