ALGEBRA OF IRRATIONAL NUMBERS

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ABSTRACT:
This article describes the Algebra of Irrational numbers. At the beginof this paper we define what an irrational number is, distinguishing it fromnumber system or rather we say from rational numbers. Some commonly known rational and irrational numbers were discussed. In order to define irrational number, we take a short glance of its history. We then seek some knowledge concerning the relation between the two sets, rational and irrational etc. We focus on three main issues:"Why rational into irrational are irrational?A short way to find inverse of irrational numbers, A small glimpse of why addition of irrational numbers might bean irrational?"We also give simple proof regarding the issues.

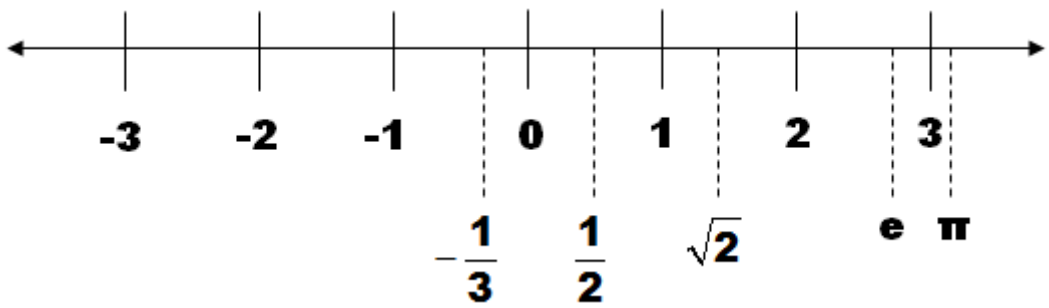
KEY WORDS: Algebra, irrational numbers, number system, rational numbers, inverse.

1.INTRODUCTION:

In the world of numbers, fundamentally starting with Natural numbers then proceeding till Real and then Complex ones, we characterize them. In Complex it is to be Real numbers and Imaginary, also in Real numbers we have Rational numbers and the numbers not being rational, which are 'Irrational numbers',where the well-known to us are Rational numbers, here helping us to know about the Irrational numbers. Let us start with some basics.

A real number is a number that can be found on the number line. These are the numbers that we normally use and apply in real-world applications.

Fig.1

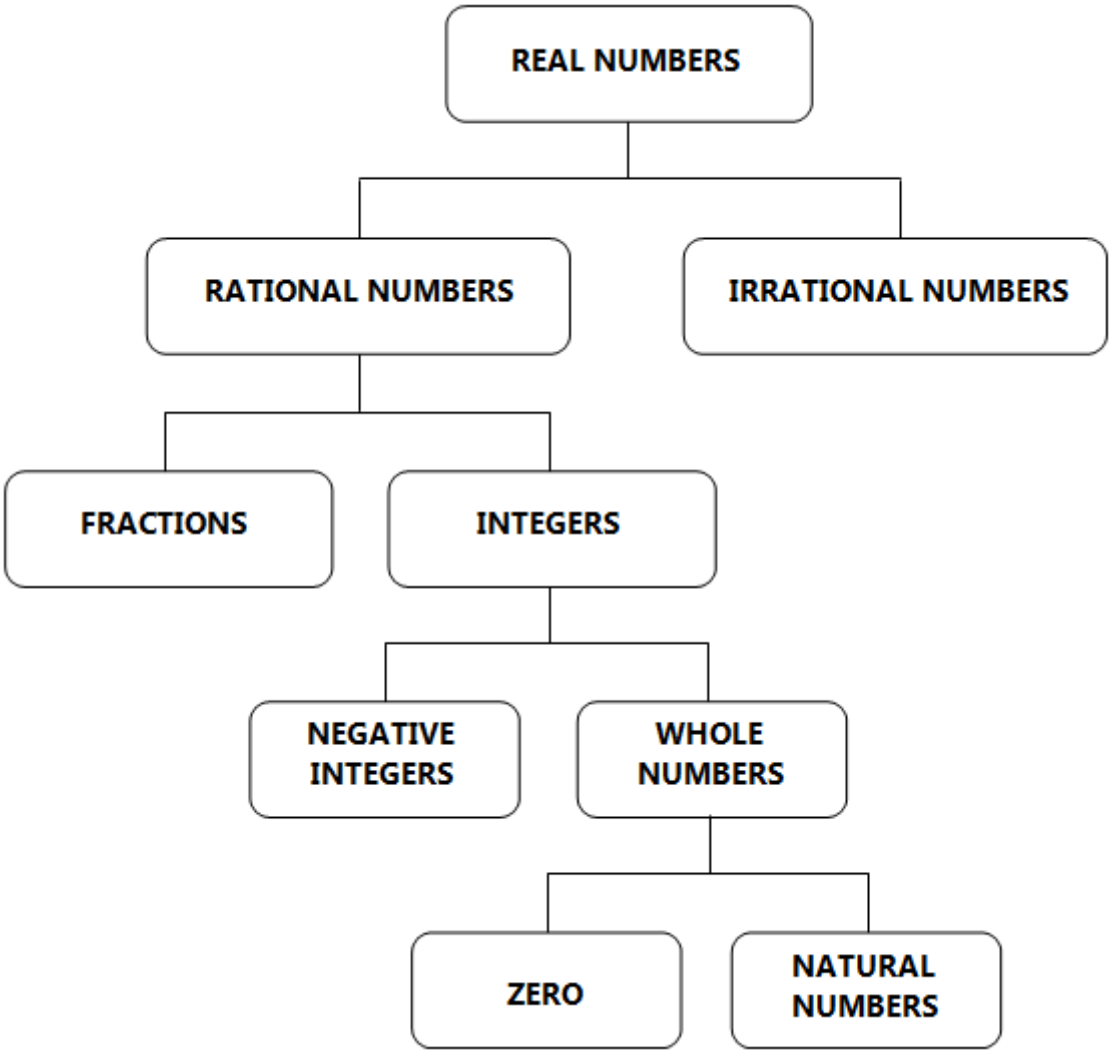


There are many types of real numbers.

- Natural Numbers
- Integers
- Fractions
- Rational numbers
- Irrational numbers
- Whole numbers

Below is a diagram of the real number system.

Fig.2



Definition
A rational number is a number that can be written as p/q, where p and q are integers.
e.g: 3, 0, -1, 5/6, 10/12, 234.45, -9.777890, etc. are rational numbers.[1]

Definition
An irrational number is a number that cannot be written as p/q with p and q integers.
e.g: π, e, are irrational numbers.[2]

Irrational numbers are non-repeating infinite decimals while rational numbers may or may not be repeating finite decimals.
For example:
Irrational numbers include $'\pi'$ which starts with 3.1415926535… and is never ending number, square roots of 2,3,7,11, etc. are all irrational numbers.
$\sqrt{2}$, $\sqrt{3}$, $\sqrt{19}$, $5+\sqrt{7}$ are all positive irrational numbers.
Similarly, $-\sqrt{3}$, $-\sqrt{5/3}$, $\sqrt{19}$, are also irrational numbers which are negative irrational numbers.
But numbers such as $\sqrt{9}$, $\sqrt{81}$, $\sqrt{25/49}$ are not irrational because 9, 81 and 25/49 are square root of 3, 9 and 5/7 respectively.
The solution of $x^2 = d$ are also irrational numbers if d is not a perfect square.
Euler's number 'e' is also an irrational number whose value is 2.71828 (approx.) and is the limit of $\left(1+\frac{1}{n}\right)^n$. it can also be calculated as sum of infinite series.

2.OVERVIEW

Since we know less about Irrational numbers, in this research paper we discussed about the behaviour of irrational numbers and some of its properties.

2.1.HISTORICAL BACKGROUND

It started with relation of measures of sides of a right angle triangle, two of each having length 1 and the largest side having length $\sqrt{2}$, that the Pythagoras theorem. Quite a problem to Pythagoras himself and his followers. Pythagoras theorem was the first proof of existence of irrational numbers.

The Vedic period in India addressed problem involving irrationals, which also suggested that Aryabhatta (5^{th}century AD) also gave the value of 'π' which upto five significant places.Aryabhatta also used the word asana(approaching), meaning approximation and incommensurable. Madhava of Sangamagrama of Kerala school of astronomy and mathematics in 14th and 16th century found infinite series for several irrational numbers. Even in the middle ages, it was allowed by Muslim mathematicians for irrationalnumber to be called as algebraic objects as development in Algebra.Book 10 of elements also classified quadratic and cubic irrationals considering irrationals as numbers.[2]

3.METHODOLOGY

In short, an Irrational number is any real number that cannot be expressed as a ratio of integers. Irrational numbers cannot be represented as terminating or repeating decimals.This paper proposes a methodology by which we seek information of irrational numbers. This research is anintellectual guide for making number mathematics more relevant to learners through geometrical representation.
The problem of defining an approximation is variation on the question "Ifyou give me an irrational number how well can I approximate it with rational numbers& how efficiently?"Duffin–Schaeffer conjecture is a possible way for the problem of approximating irrationals using rational numbers by just following simple steps and for any approximation working provided you give space for a little error.[3] The first step is to choose an infinitely long list of denominators. Secondly, you need to choose how closely you would like to approximate the irrational numbers? That is, you need to choose error term.
Regardless of what irrational numbers you choose there are infinitely many different pairs, where we have efficient approximation that is error term is pretty close by comparison when we are just considering a decimal expansion, we would just have an error term 1 over the denominator, whatever we choose for approximation. Almost everything can be approximated or almost nothing can be approximated in the way you want regardless of how you choose your denominator or how you choose your efficiency of approximation.
So, choose your denominators say, $q_1, q_2, q_3 \ldots$ and now choose tolerable errors so that how efficient you want to do for each denominator say, $e_1, e_2, e_3 \ldots$
Then, can you approximate your rational number α by denominator from your favourite set with error term of your choice

$$i.e. \left| \alpha - \frac{a}{q_i} \right| \leq e_i \quad [4]$$

Duffin–Schaeffer conjecture also provides us with the outcome that any irrational number can be approximated in this fashion.
As we get approximating sequences for irrational numbers let us move tosome more details.

4.ISSUES:

1.) Why rational number into irrational number are irrational number?

Proof:

Let 'x' be irrational.
Let 'y' be rational,

$\Rightarrow y = \frac{p}{q}$, for $(p,q)=1$

Claim:

$(\text{irrational No.}) \times (\text{rational No.}) = (\text{irrational No.})$

$x \cdot y = irrational$

On contrary assume that it is rational.

i.e. $x \cdot y = \frac{p'}{q'}$ for $(p',q')=1$

$\Rightarrow x \cdot \frac{p}{q} = \frac{p'}{q'}$

$\Rightarrow x = \frac{p'}{q'} \cdot \frac{q}{p}$

$\Rightarrow x = \frac{p_1}{q_1}$

$\therefore x$ is rational number,which is contradiction to our assumption that x is irrational.

$\Rightarrow$ rational number into irrational number is irrational number.

2.) If 'x' is an irrational, (x)-1 is also an irrational.

Claim: If $'x'$ is an irrational, $(x)^{-1}$ is also an irrational.

Proof:

On contrary suppose x^{-1} is a rational implies

$x^{-1} = \frac{p}{q}$

$\Rightarrow \quad x \cdot x^{-1} = x \cdot \frac{p}{q}$

$\Rightarrow 1 = x \cdot \frac{p}{q}$

$\Rightarrow x = \frac{q}{p}$

$\Rightarrow x$ is a rational number , which is contradiction

$\therefore x^{-1}$ needs to be an <u>irrational</u>.

3.) Inverse of an irrational number:

As a matter of fact, we know measure of the rational numbers is '0', which immediately gives the measure of whole of $\mathbb{R}$to the irrationals. Irrationals the so called rogues, or transcendental or the going on has given us area of circle, smooth functions, and what not they really worth to be known about.

Let just try to know them using their rational approximations.

Let 'x' be an irrational number. Let $(x_n)_{n=1}^{\infty}$ be a sequence of rational numbers converging to 'x'.

Claim: Given $(x_n)_{n=1}^{\infty} \to x$

To Prove $\left(\frac{1}{x_n}\right)_{n=1}^{\infty} \to \frac{1}{x}$

Proof:

Given that $(x_n)_{n=1}^{\infty} \to x$

$\Rightarrow$ for given $\varepsilon > 0$, $\exists N$s.t $N \in \mathbb{N}$

$$|x - x_n| = |x_n - x| < \varepsilon$$

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n \cdot x}\right|$$

$$= \frac{|x - x_n|}{|x_n \cdot x|}$$

$$= \frac{|x - x_n|}{|x|^2}$$ As we can choose $(x_n)_{n-1}^{\infty} \to x$

Now, $1 < |x_n|^2 < |x|^2$

$$1 > \frac{1}{|x_n|^2} > \frac{1}{|x|^2}$$

$$1 > \frac{1}{|x|^2}$$

$$\therefore \left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x - x_n|}{|x|^2} < 1 \cdot |x - x_n| < \varepsilon$$

$$\Rightarrow \left|\frac{1}{x_n} - \frac{1}{x}\right| < \varepsilon$$ for given $\varepsilon > 0$, $n \geq N$

$$\left(\frac{1}{x_n}\right)_{n=1}^{\infty} \to \frac{1}{x}$$ For $1 < |x|$

If $1 > |x|$ then the above proof does not work.

4.) If x is an irrational number then $\sqrt{x}$ is also irrational.

$\rightarrow$ If not then $\sqrt{x} = \frac{p}{q}$, for $(p,q)=1$

On squaring both sides, we get,

$$x = \frac{p^2}{q^2}$$

$\Rightarrow$ x is a rational

which is a contradiction

$\therefore \sqrt{x}$ **has to be an irrational number.**

For example, $\sqrt{\sqrt{2}}$ as we know now that the number $\sqrt{2}$ is an irrational number let us proof that is also irrational number.

On contrary let $\sqrt{\sqrt{2}}$ be rational number,

$$\Rightarrow \sqrt{\sqrt{2}} = \frac{p}{q}, \qquad \text{for } (p,q)=1$$

On squaring both sides, we get,

$$\sqrt{2} = \frac{p^2}{q^2}$$

$\Rightarrow \sqrt{2}$ is a rational number.

which is a contradiction

$\therefore \sqrt{\sqrt{2}}$ **is an irrational number.**

5.) Addition of irrational numbers:

It is not known whether or not $e+\pi$ is irrational nor known whether or not $e \cdot \pi$ is irrational. Since, e and π are transcendental, there is no polynomial with wholly rational coefficient to have e or π as a root.

$\therefore$ Polynomial,

$$(x-e)(x-\pi)=0$$

$$\Rightarrow x^2-(e+\pi)x+e\cdot\pi=0$$

must have atleast one irrational coefficient. For a while, let us consider addition of irrational e and π that too by approximating sequences:

Table 1

Sequence	**Approximation of** e	**Approximation of** π	**Approximation of** $e+\pi$
a_1	2.7	3.1	5.8
a_2	2.71	3.14	5.85
a_3	2.718	3.141	5.859
.	.	.	.
.	.	.	.
.	.	.	.
a_{10}	2.71828 18284	3.14159 26535	5.85987 44819

So, intuitively if the approximating sequence for $e+\pi$ does have non-repeating digits in the fractional part then it is irrational or if it has repeating pattern in the fractional part then it is a rational one.

5.Conclusion:

As far irrational numbers are considered we saw their algebra, their approximating sequences, sequences converging to inverse for some irrational numbers. Lastly, we also saw a rough idea of might be an irrational number which can be an intuitive idea for addition of some transcendental numbers being irrational.

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