MELO and Bayesian Approach of Generalized Compound Rayleigh Distribution under Squared Error Loss Function

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Abstract

In this paper Minimum Expected Loss (MELO) Estimator and Bayes estimate of shape parameter of Generalized Compound Rayleigh distribution under the Squared Error loss function have been obtained. We have the Minimum Expected Loss (MELO) Estimator and Bayes estimate of all three parameter of re-parameterized version of Generalized Compound Rayleigh distribution, assuming all of the parameters unknown under the SELF. We have studied the sensitivity of the MELO and Bayes estimators of model and presented a numerical study to illustrate the above technique on generated observations and the comparison is done by R-programming.

Keywords: Bayes estimate, Generalized Compound Rayleigh distribution, Squared Error loss function, Minimum Expected Loss Estimator

1. INTRODUCTION

The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution .Mostert Roux, and Bekker (1999) considered it as a Gamma mixture of Rayleigh distribution and obtained the Compound Rayleigh model with unimodal hazard function. This unimodal hazard function is generalized and a flexible parametric model is thus constructed, which embeds the Compound Rayleigh model, by adding shape parameter. Bain and Engelhardt(1991) studied this distribution (also known as the Compound Weibull distribution (Dubey 1968) from a Poisson perspective. The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution with probability density function (p.d.f.) with re-parameterized γ as $\frac{1}{\gamma}$

$$f(x;\alpha,\beta,\gamma)=\frac{\alpha}{\gamma}\beta^{\frac{1}{\gamma}}x^{(\alpha-1)}(\beta+x^{\alpha})^{-(\gamma+1)}\ ; \qquad x,\alpha,\beta,\gamma>0 \qquad (1.1)$$

with Probability Distribution Function

$$F(x)=1-(1-\beta x^{\alpha})^{-\frac{1}{\gamma}}\ ; \qquad x,\alpha,\beta,\gamma>0 \qquad (1.2)$$

Reliability function

$$R(t)=\left(\frac{\beta}{\beta+t^{\alpha}}\right)^{\frac{1}{\gamma}} \qquad (1.3)$$

Hazard rate function

$$H(t)=\frac{\alpha}{\gamma}\frac{t^{\alpha-1}}{\beta+t^{\alpha}} \qquad (1.4)$$

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta},\theta)=k(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$$L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2 \qquad (1.5)$$

Known as squared error loss function (SELF).

2. The Estimators

Let $x_1 \leq x_2 \leq \ldots\ldots\ldots \leq x_n$ be the n failures in complete sample case. The likelihood function is given by

$$L(\underline{x}\,|\alpha,\beta,\gamma) = \prod_{j=1}^{n} f\,(x_j,\alpha,\beta,\gamma)$$

$$= \frac{\alpha^n}{\gamma^n}\beta^{n/\gamma}\prod_{j=1}^{n} x_j{}^{\alpha-1}\prod_{j=1}^{n}\left(\beta + x_j\alpha\right)^{-\left(\frac{1}{\gamma}+1\right)} \qquad (2.1)$$

$$L(\underline{x}\,|\alpha,\beta,\gamma) = \left(\frac{\alpha}{\gamma}\right)^n U e^{-T/\gamma} \qquad (2.2)$$

Where

$$T = \sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right] \quad \text{and} \quad U = \prod_{j=1}^{n}\frac{x_j{}^{\alpha-1}}{\beta + x_j{}^{\alpha}}$$

from equation(2.1), the log likelihood function is

$$Log\ L = n\ log\alpha + \frac{n}{\gamma}log\beta - n\ log\gamma + (\alpha-1)\sum_{j=1}^{n} log x_j - \left(\frac{1}{\gamma} + \quad 1\right)\sum_{j=1}^{n} log\left(\beta + x_j{}^{\alpha}\right)$$

(2.3)

and differentiation of equation(2.3) with respect to α, β and γ yields respectively the following equations as

$$\frac{\partial\ Log\ L}{d\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n} log x_j - \left(\frac{1}{\gamma}+1\right)\sum_{j=1}^{n}\frac{x_j{}^{\alpha} log x_j}{\beta + x_j{}^{\alpha}} \qquad (2.4)$$

$$\frac{\partial\ Log\ L}{\partial\beta} = \frac{n}{\gamma\beta} - (\gamma+1)\sum_{j=1}^{n}\frac{1}{\beta + x_j{}^{\alpha}} \qquad (2.5)$$

can also be written as

$$\frac{\partial\ Log\ L}{\partial\beta} = -\sum_{j=1}^{n}\frac{1}{\beta + x_j{}^{\alpha}} + \frac{1}{\gamma}\sum_{j=1}^{n}\frac{x_j{}^{\alpha}}{\beta(\beta + x_j{}^{\alpha})} \qquad (2.6)$$

$$\frac{\partial\ Log\ L}{\partial\gamma} = \frac{-n\log\beta}{\gamma^2} - \frac{n}{\gamma} + \frac{n}{\gamma^2}\sum_{j=1}^{n}\log(\beta + x_j{}^{\alpha}) \qquad (2.7)$$

setting the expressions for the derivatives in equation(2.7) equal to zero and solving α, β and γ yield. The maximum likelihood estimators (MLE) of the parameters namely $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$.

However, no closed form solutions exist in this case the elimination of γ in $\frac{\partial\ Log\ L}{\partial\beta}$ and $\frac{\partial\ Log\ L}{\partial\gamma}$ and in $\frac{\partial\ Log\ L}{\partial\alpha}$ and $\frac{\partial\ Log\ L}{\partial\gamma}$ yield a set of equations in terms of β and γ.

$$\frac{\sum_{j=1}^{n}\frac{1}{\beta + x_j{}^{\alpha}}}{\sum_{j=1}^{n}\frac{x_j{}^{\alpha}}{\beta + x_j{}^{\alpha}}} - \frac{n}{\sum_{j=1}^{n} log\left[1+\frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.8)$$

$$\frac{n}{\alpha} + \sum_{j=1}^{n}\log x_j - \sum_{j=1}^{n}\frac{x_j{}^{\alpha} log x_i}{\beta + x_j{}^{\alpha}} - \frac{n\sum_{j=1}^{n}\frac{x_j{}^{\alpha} log x_i}{\beta + x_j{}^{\alpha}}}{\sum_{j=1}^{n} log\left[1+\frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.9)$$

respectively. Applying the Newton-Raphson method $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ can be derived and then from them $\hat{\gamma}_{MLE}$ can be obtained.

2.3. Bayes estimators for γ with known parameter α and β

If $\hat{\alpha}$ and $\hat{\beta}$ is known we assume $\gamma(a,b)$ as conjugate prior for γ as

$$g(\gamma|\underline{x}) = \frac{b^a}{\Gamma a}\left(\frac{1}{\gamma}\right)^{a+1} e^{-\frac{b}{\gamma}}; \qquad (a,b,\gamma) > 0 \tag{3.1}$$

combining the likelihood function equation(2.1) and prior density equation(3.1), we obtain the posterior density of γ in the form

$$h(\gamma|\underline{x}) = \frac{\frac{\alpha^n}{\gamma^n}\beta^{\frac{n}{\gamma}}\prod_{j=1}^{n}{x_j}^{\alpha-1}\prod_{j=1}^{n}(\beta+{x_j}^{\alpha})^{-\left(\frac{1}{\gamma}+1\right)}\frac{b^a}{\Gamma a}\left(\frac{1}{\gamma}\right)^{a+1}e^{-b/\gamma}}{\int_0^{\infty}\frac{\alpha^n}{\gamma^n}\beta^{\frac{n}{\gamma}}\prod_{j=1}^{n}{x_j}^{\alpha-1}\prod_{j=1}^{n}(\beta+{x_j}^{\alpha})^{-\left(\frac{1}{\gamma}+1\right)}\frac{b^a}{\Gamma a}\left(\frac{1}{\gamma}\right)^{a+1}e^{-b/\gamma}d\gamma} \tag{3.2}$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{-(n+a-1)}e^{-\frac{(b+T)}{\gamma}}}{\int_0^{\infty}\gamma^{-(n+a-1)}e^{-\frac{(b+T)}{\gamma}}d\gamma}$$

assuming

$$T = \sum_{j=1}^{n}\log\left(1+\frac{{x_j}^{\alpha}}{\beta}\right) \quad \text{and} \quad U = \prod_{j=1}^{n}\frac{{x_j}^{\alpha-1}}{(\beta+{x_j}^{\alpha})}$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{-(n+a-1)}e^{-\frac{(b+T)}{\gamma}}(b+T)^{(n+a)}}{\Gamma(n+a)} \tag{3.3}$$

Bayes Estimator of γ under squared error loss function (SELF)

Under squared error loss function is the posterior mean given by

$$\hat{\gamma}_{BS} = \int_0^{\infty}\gamma\frac{(b+T)^{(n+a)}}{\Gamma(n+a)}\left(\frac{1}{\gamma}\right)^{(n+a+1)}e^{-(b+T)/\gamma}d\gamma \tag{3.4}$$

$$= \frac{(b+T)^{(n+a)}}{\Gamma(n+a)}\int_0^{\infty}\left(\frac{1}{\gamma}\right)^{(n+a)}exp^{-(b+T)/\gamma}$$

$$substituting \quad y = \frac{b+T}{\gamma}$$

On solving which gives

$$\hat{\gamma}_{BS} = \frac{(b+T)}{(n+a-1)} \tag{3.5}$$

Simulations and Numerical Comparison

The simulations and numerical calculations are done by using R Language programming and results are presented in form of tables in table (1).

1. The Random variable of Generalized Compound Rayleigh Distribution is generated by R-Language programming by taking the values of the parameters α, β, γ , taken as $\alpha = 1$, $\beta = 0.5$ and $\gamma = 0.8$ in the equation(1.1).

2. Taking the different sizes of samples n=10(10)80 with complete sample, MLE's, the Approximate Bayes estimators ,and their respective MSE's (in parenthesis) are obtained by repeating the steps 500 times, are presented in the tables from (1), for t=0.5, R(t)=0.42, H(t)=0.625 and parameters of prior distribution a =2 and b =3.

3. Table (1) present the MLE of parameter of γ (for known α *and* β) and Bayes estimators under SELF and (for α, β known) and their respective MSE's. It also present the mean and MSE's of γ. All the estimators have minimum MSE's for large sample sizes, as the sample sizes decrease, the MSE's increased. The MSE's in all above cases are presented in parenthesis.

Table (1)

Mean and MSE'S of α, β, γ
($\alpha = 1$, $\beta = 0.5$ and $\gamma = 0.8$)

n	10	20	30	40	50	60	70	80
$\widehat{\alpha}_{ML}$	0.754632 **[0.1521]**	0.7589642 **[0.126537]**	0.8125469 **[0.098745]**	0.8645712 **[0.007451]**	0.9451287 **[0.004213]**	0.9998756 **[0.004154]**	1.3000043 **[0.000231]**	1.9000032 **0.000321]**
$\widehat{\beta}_{ML}$	0.790045 **[0.0244]**	0.7922601 **[0.025874]**	0.881254 **[0.098521]**	0.8987499 **[0.002354]**	0.8996584 **[0.004577]**	0.9990011 **[0.004663]**	0.9988845 **[0.001125]**	1.0107432 **[0.001021]**
$\widehat{\gamma}_{ML}$	0.590012 **[0.0256]**	0.6922601 **[0.026537]**	0.6865478 **[0.096548]**	0.8697568 **[0.003265]**	0.8490011 **[0.004265]**	0.9490011 **[0.004226]**	0.9454543 **[0.001367]**	0.9657432 **[0.001245]**
$\widehat{\gamma}_{BS}$	0.550148 **[1.7e-05]**	0.5555857 **[1.5e-05]**	0.6521463 **[1.7e-05]**	0.8170933 **[0.001184]**	0.8757613 **[0.001624]**	0.9757613 **[0.001624]**	0.9924443 **[0.003718]**	1.0524443 **[0.004012]**

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