COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS SATISFYING *CLCS*-PROPERTY

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Abstract. In this article we have established some fixed point results using common limit converging in the subset *CLCS*-property defined by [13]. Our result extend and generalize the result established earlier by various authors such as [13] and [19].

1. Introduction

In 2011, Azam et al. [1] introduce the notion of of new space called complex valued metric space and establishes existence of fixed point theorems under the contraction condition.The theorems proved by Azam et al. [1] and Bhatt et.al. [14] uses the rational inequality in a complex valued metric space as contractive condition.

Theorem 1.1. *([1]). Let (X,d) be a complete complex valued metric space and $S,T : X \to X$, If S and T satisfy*

$$d(Sx,Ty) \preceq \lambda d(x,y) + \frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)} \qquad (1.1)$$

for all $x,y \in X$, where λ,μ are nonnegative reals with $\lambda + \mu < 1$. Then S and T have a common fixed point.

In 2012, Rouzkard et.al. [7] established some common fixed point theorems satisfying certain rational expressions in Complex valued metric space.

Theorem 1.2. *([7]). If S and T are self mappings defined on a complex valued metric space (X,d) satisfying the condition,*

$$d(Sx,Ty) \preceq \lambda d(x,y) + \frac{\mu d(x,Sx)d(y,Ty) + \gamma d(y,Sx)d(x,Ty)}{1+d(x,y)} \qquad (1.2)$$

for all $x,y \in X$, where λ,μ,γ are nonnegative reals with $\lambda + \mu + \gamma < 1$. Then S and T have a unique common fixed point.

Later on Sintunavarat et.al. [8] extend and improve the condition of contraction of theorem (1.1) from the constant of contraction to some control functions and establish the common fixed point theorems which are more general than the result of [1] and also give the results for weakly compatible mappings in complex valued metric spaces.

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Theorem 1.3. *([8]) Let (X,d) be a complete complex valued metric space and $S,T : X \to X$, if there exists a mappings $\Lambda,\Xi : X \to [0,1)$*

(1) $\Lambda(Sx) \le \Lambda(x)$ *and* $\Xi(Sx) \le \Xi(x)$;

(2) $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$;

(3) $(\Lambda + \Xi)(x) < 1$;

$$d(Sx,Ty) \preceq \Lambda(x)d(x,y) + \frac{\Xi(x)d(x,Sx)d(y,Ty)}{1+d(x,y)} \qquad (1.3)$$

for all x,y ∈ X, where λ,μ are nonnegative reals with λ + μ < 1. Then S and T have a common fixed point.

The study of existence of common fixed point grown from weakly commutativity [8] to compatibility [15] and weakly compatibility [11].
Similarly, the non-commutativity of mappings grown from non compatibility [11] to property (E.A.) [16].

In 2014 Hakawadiya et.al. [4], Proved common fixed point theorems for six self mappings as follows,

Theorem 1.4. *Let (X,d) be a complex valued metric space and A,B,D,M,S and T be six self mappings in X satisfying the condition:*

(1) *S(X) ⊂ BD(X) and T(X) ⊂ AM(X);*

(2) *(AM,S) and (BD,T) are commuting pairs;*

(3) *The pair (AM,S) and (BD,T) are weakly compatible;*

(4) *For each x,y ∈ X and x 6= y,;*

(i): *If d(Ty,AMx)+d(Sx,BDy)+d(BDy,AMx) = 0. then d(Sx,Ty) = 0;*

(ii): *If d(Ty,AMx) + d(Sx,BDy) + d(BDy,AMx) 6= 0 than following identity holds;*

$$\begin{aligned} d(Sx,Ty) \preceq\ & \alpha\left(\frac{d(AMx,Sx)+d(BDy,Ty)d(AMx,Sx)}{1+d(Sx,Ty)}\right) \\ &+ \beta max\{d(AMx,BDy)), d(AMx,Sx), d(BDy,Sx)\} \\ &+ \gamma\{d(BDy,Ty)+d(Ty,AMx)+d(Sx,BDy)\} \\ &+ \eta\left(\frac{d(Ty,BDy)d(Sx,AMx)}{d(Ty,AMx)+d(Sx,BDy)+d(BDy,AMx)}\right) \end{aligned}$$

Where α+β +2γ +η < 1. Then AM,BD,S and T have a unique common fixed point.

The concept of (E.A.) property allows us to replace the completeness requirement of the space by a more natural condition of closeness of range. Pathak, Lopez and verma [17] proved a common fixed point theorem in metric space or an integral type implicit relation using the property (E.A.). By using the property of (E.A.) Sintunavarat [9] introduced the concept of 'common limit range property' or (CLR)- property, for a pair of mappings.

In 2019, Verma et.al. [18] introduced the concept of common limit converging in the subset or (CLCS)- property and proved following common fixed point theorem for two pairs of weakly compatible mappings in a complex valued metric space.

Theorem 1.5. *Let (X,d) be a complex valued metric space and A,B,S and T: X → X be four self mappings satisfying:*

(1) *(A,S) satisfy (CLCS) property in T(X) or B(X), and B(T) satisfy (CLCS) property in S(X) or A(X).*

(2) $d(Ax, By) \preceq \phi(max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(By,Sx), d(Ax,Ty)\})$

for all x,y ∈ X. If (A,S) and (B,T) are weakly compatible then mappings A,B,S and T have a unique common fixed point in X.

2. preliminaries

An ordinary metric d is a real-valued function from a set $X \times X$ into R, where X is a nonempty set. That is, $d : X \times X \to$ R. A complex number $z \in$ C is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Thus a complex valued metric d is a function from a set $X \times X$ into C, where X is a nonempty set and C is the set of complex number.

Let C be the set of complex numbers and $z_1, z_2 \in$ C. We define a partial order$'\preceq'$ on $\mathbb{C}$ as follows:

(A): Two complex number z_1, z_2 such that $z_1 \preceq z_2 \iff Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$.

Thus $z_1 \preceq z_2$ if one of the following holds:

(C1): $Re(z_1) = Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;

(C2): $Re(z_1) < Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;

(C3): $Re(z_1) = Re(z_2)$ *and* $Im(z_1) < Im(z_2)$; **(C4):** $Re(z_1) < Re(z_2)$ *and* $Im(z_1)$

$< Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of ($C2$), ($C3$), and ($C4$) is satisfied and we will write $z_1 \prec z_2$ if only ($C4$) is satisfied.

(B): It follows that,

(1) $0 \preceq z_1 \precnsim z_2 \ implies |z_1| < |z_2|$;

(2) $z_1 \preceq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$; (3) $0 \preceq z_1 \preceq z_2$ implies $|z_1| \le |z_2|$;

(4) if $a,b \in$ R, $0 \le a \le b$ and $z_1 \preceq z_2$, then $az_1 \preceq bz_2$ for all $z_1, z_2 \in$ C.

Definition 2.1. [1]. Let X be a nonempty set. A mapping $d : X \times X \to$ C is called a complex valued metric on X if the following conditions satisfied :

1. $0 \le d(x,y)$*, for all x,y ∈ X and* $d(x,y) = 0$ *if and only if* $x = y$;
2. $d(x,y) = d(y,x)$ *for all x,y ∈ X*;
3. $d(x,y) \le d(x,z) + (z,y)$*, for all x,y,z ∈ X.*

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.2. Let $X =$ C, Define the mapping $d : X \times X \to$ C by

$$d(z_1,z_2) = e^{-il}|z_1 - z_2|,$$

where $z_1, z_2 \in$ C. and $l \in$ R Then (X,d) is a complex valued metric space.

Definition 2.3. [8] Let (X,d) be a complete complex valued metric space, $\{x_n\}$ be a sequence in X,

1. *A point x ∈ X is called interior point of a set A ⊆ X whenever there exists* $0 < r \in C$ *such that* $B(x,r) := \{y \in X | d(x,y) < r\} \subseteq A$.

(2) *A point* $x \in X$ *is called a limit point of* Z *whenever for every* $0 < r \in C, \{B(x,r) \cap (A - X)\} 6= \varphi$

(3) *A subset* $A \subseteq X$ *is called open whenever each element of* A *is an interior point of* A

(4) *A subset* $A \subseteq X$ *is called closed whenever each limit point of* A *belongs to* A*.*

(5) *A sub − basis for a Hausdorff topology* τ *on* X *is a family*

Definition 2.4. [8]. Let (X,d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in C$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$. $d(x_m,x) < c$, then $\{x_n\}$ is said to *convergent*, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. we denote this by

$$\lim_{n\to\infty} x_n = x$$

.

(2) If for every $c \in C$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x_{n+m}) < c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete valued metric space.

Remark 2.5. (1) If A^0 is the set of limit points of ‘A‘ and there exist ${}^0u^0$ such that $0 < u < z$ for each $z \in A^0$ then $u = 0$,

(2) If $z \leq \lambda z$ and $0 \leq \lambda < 1$, then $z = 0$.

Lemma 2.6. [1] *Let* (X,d) *be a complex valued metric space and* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *converges to x if and only if* $|d(x_n),x| \to 0$ *as* $n \to \infty$*.*

Lemma 2.7. [1] *Let* (X,d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *is a cauchy sequence if and only if* $|d(x_n,x_{n+m})| \to 0$ *as* $n \to \infty$*, where* $m,n \in \mathbb{N}$

Definition 2.8. [1] Let P and Q be self mappings of a nonempty set X

(1) A point $x \in X$ is said to be a fixed point of P if $Px = x$.

(2) A point $x \in X$ is said to be a coincidence point of P an Q if $Px = Qx$ and we shall called $w = Px = Qx$ that a point of coincidence of P and Q.

(3) A point $x \in X$ is said to be a common fixed point of P and Q if $x = Px = Qx$.

In 1976, Jungck [20] introduced concept of common mappings as follows:

Definition 2.9. Let X be a non-empty set. The mappings P and Q are commuting if $PQx = QPx$, for all$x \in X$.

In 1986, [15] introduced the more generalized commuting mappings in metric spaces, called compatible mappings., which also are more general than the concept of weakly commuting mappings as follows:

Definition 2.10. Let P and Q be mappings from a metric space (X,d) into itself. The mappings P and Q are said to be compatible if

$$\lim_{n\to\infty} d(PQx_n,QPx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Px_n = z$$

for some $z \in X$.

Example 2.11. Let $X = \mathbb{C}$ we define complex-metric $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2) = e^{-il}|z_1 - z_2|$, where $a \in [0, \frac{\pi}{2}]$. Then (X,d) is a complex valued metric space. Suppose $P,Q : X \to X$ be defines as

$Pz = 4e^{\frac{\pi}{6}}$, $if \quad Re(z) \neq 0$, $\quad Pz = 2e^{\frac{\pi}{4}}$, $if \quad Re(z)$ = 0 and $Qz = 4e_{-6}$, if $Re(z)$ 6= 0, Qz = $3e_{-3}$, if $Re(z) = 0$

Then one can observe that $Pz = Qz = 4e^{\frac{\pi}{6}}$, when $Re(z)$ 6= 0, and so $PQz = QPz = 4e^{\frac{\pi}{6}}$. Hence (P,Q) is weakly compatible at all $z \in \mathbb{C}$ with $Re(z)$ 6= 0.

Remark 2.12. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converse are not necessarily true.

Example 2.13. Let $X = [0,2]$ with usual metric d where $d(x,y) = |x - y|$ for all x and y in X. We define Px and Qx as follows

$$Px = \begin{cases} x, & \text{if } x \in [0,1) \\ 2, & \text{if } x \in [1,2] \end{cases}$$

$$Qx = \begin{cases} 2 - x, & \text{if } x \in [0,1) \\ 2, & \text{if } x \in [1,2] \end{cases}$$

by choosing $x_k = 1 - \frac{1}{k}$ then $Px_k = 1 - \frac{1}{k}$ and $Qx_k = 1 + \frac{1}{k}\ \forall k \geq 1$ one can easily show the commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converse are not necessarily true.

In 1996, Jungck introduced the concept of weakly compatible mappings as follows:

Definition 2.14. Let P and Q be self mappings of a non empty set X. The mappings P and Q are weakly compatible if $PQx = QPx$ whenever $Px = Qx$.

We can see an example to show that there exist weakly compatible mappings which are not compatible mappings in Djoudi and Nisse.

The following lemma proved by Haghi et al. is useful for our main results:

Example 2.15. Let $X = [2,20]$ with usual metric d where $d(x,y) = |x - y|$ for all x and y in X.We define Px and Qx as follows

$$Px = \begin{cases} 2, & \text{if } x = [0,1) \\ 13 + x, & \text{if } 2 \geq x \leq 5 \\ x - 3, & \text{if } x > 5 \end{cases}$$

$$Qx = \begin{cases} 2, & \text{if } x \in 2 \cup (5,20] \\ 8, & \text{if } x \in (2,5] \end{cases}$$

by choosing $x_k = 5 + \frac{1}{k}$ for all $k \geq 1$. The map P and Q are compatible maps.

Definition 2.16. [18] Suppose that (X,d) is a metric space and $P,Q : X \to X$. Two mappings P and Q are said to satisfy the common limit in the range of Q property, in short, (CLR_Q)-property if:

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = Qx \qquad (2.1)$$

for some $x \in X$.

In a complex-valued metric space (X,d), in 2.16 will be same but the space X will be a complex valued metric space.

Remark 2.17. [18] If the mapping $P,Q : X \to X$ satisfy (CLR_Q)-property, then it also satisfy (CLR_P)-property and vice versa. Following are are the examples that justify our remark

Example 2.18. Let $X = \mathbb{C}$ and d be any complex valued metric space on X. We define $P,Q : X \to X$ by $Pz = z + 2i$ and $Qz = 3z\ \forall z \in X$. Consider a sequence

$\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ in X, then

$$\lim_{n\to\infty} Pz_n = \lim_{n\to\infty} z_n + 2i = \lim_{n\to\infty}(i + \frac{1}{n}) + 2i = 3i = 3(0+i) = Qz$$

and

$$\lim_{n\to\infty} Qz_n = \lim_{n\to\infty} 3(i + \frac{1}{n}) = 3i = 3(0+i) = Qz.$$

Hence, the pair (P,Q) satisfies (CLR_Q)-property in X with $z = 0 + i \in X$.

Example 2.19. [18] Let X = C and $d(z_1,z_2) = e^{il}|z_1-z_2|$ be any complex valued metric on X. We define $P,Q : X \to X$ by $Pz = 2z - 4$ and $Qz = z + 2i$, $\forall z \in X$.

Consider a sequence $\{z_n\} = \{4 + 2i + \frac{1}{n}\}_{n\geq 1}$ in X, then

$\lim_{n\to\infty} Pz_n = \lim_{n\to\infty} 2z_n - 4 = 4 + 4i = \lim_{n\to\infty} Qz_n = \lim_{n\to\infty} z_n + 2i = 4 + 4i = Q(4 + 2i)$

Hence the pair (P,Q) satisfies property (CLR_Q) with $z = 4 + 2i \in X$.By above it follows that, (P,Q) satisfy (CLR_P) property also in X.

In definition 2.16 the notion of (CLR) property does not require the condition of closeness of the range (sub)space but the common limit $^0t^0$ goes to different sets (in Px for CLR_P and in Qx for CLR_Q).

By unifying above definition of CLR_P and CLR_Q and generalizing the $(E.A.)$ property Verma et.al.[18] defines COMMON LIMIT CONVERGING IN THE SUBSET $(CLCS)$ - property as follows

Definition 2.20. [18] Suppose that (X,d) be a complex valued metric space and $P,Q : X \to X$. Let $Y \subseteq X$ the mapping P,Q are said to satisfy the property of Common Limit in the Subset $(CLCS)$ in Y, if there exist a sequence $\{z_n\}$ in X such that,

$$\lim_{n\to\infty} Pz_n = \lim_{n\to\infty} Qz_n \in Y \qquad (2.2)$$

for some sequence $\{z_n\}$ in X.

In this article we shall use 2.20 stated by [18] in our main results.

3. Main results

Let $\psi : C \to C$ be a continuous function with the partial order relation $^0\prec^0$ in C such that $\psi(0) = 0$ and $|\psi(kp)| < k|\psi(p)| < k|(p)|$, $\forall p \in C$.

Theorem 3.1. *Let (X,d) be a complex valued metric space. Suppose that the mapping P,Q,R and S be four self maps on X satisfying the following condition*

(1) (P,R) satisfy $(CLCS)$ - property in $S(X)$ or $Q(X)$ and (Q,S) satisfy $(CLCS)$ - property in $P(X)$ or $R(X)$

(2)

$$\begin{aligned} d(Px,Qy) \preceq\ & \psi\{\beta_1 d(Rx,Sy) + \beta_2 d(Px,Rx) \\ & + \beta_3 d(Sy,Qy) + \beta_4 \frac{d(Rx,Qy) + d(Px,Sy)}{2} \\ & + \beta_5 \frac{d(Rx,Px)d(Sy,Qy)}{1 + d(x,y)}\} \end{aligned} \qquad (3.1)$$

$\forall\ x,y \in X$, where $\sum_{i=1}^{i=5} \beta_i < 1$ and all $\beta_i > .0$If (P,R) and (Q,S) are weakly compatible then mappings P,Q,R and S have a unique common fixed point in X.

Proof

We take condition (1) *of the above statement,*

Case(I): Suppose that the pair (P,R) *satisfy* $(CLCS)$ *- property in* $S(X)$*. Then according to the definition 2.20, there exist* $\{x_n\}$ *in X such that*

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n \in S(X).$$

So that there exist $t \in S(X)$*. Such that* $t = S\nu$ *for some* $\nu \in X$*, where*

$$t = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n$$

we claim that $Q\nu = t$*, i.e.* $d(t,Q\nu) = 0$*.*

If not then putting $x = x_n$ *and* $y = \nu$ *in* 3.9 *we have,*

$$\begin{aligned} d(Px_n, Q\nu) \preceq\ & \psi\{\beta_1 d(Rx_n, S\nu) + \beta_2 d(Px_n, Rx_n) \\ & + \beta_3 d(S\nu, Q\nu) + \beta_4 \frac{d(Rx_n, Q\nu) + \beta_5 d(Px_n, S\nu)}{2} \\ & + \beta_5 \frac{d(Rx_n, Px_n) d(Rx_n, Q\nu)}{1 + d(x_n, \nu)}\} \end{aligned} \qquad (3.2)$$

Letting $n \to \infty$

$$\begin{aligned} d(t, Q\nu) \preceq\ & \psi\{\beta_1 d(t,t) + \beta_2 d(t,t) \\ & + \beta_3 d(t, Q\nu) + \beta_4 \frac{d(t, Q\nu) + \beta_5 d(t,t)}{2} \\ & + \beta_5 \frac{d(t,t) d(t, Q\nu)}{1 + d(x_n, \nu)}\} \end{aligned}$$

$$d(t, Q\nu) \preceq \psi\{\beta_3 d(t, Q\nu) + \beta_4 \frac{d(t, Q\nu)}{2}\}$$

$$\begin{aligned} |d(t, Q\nu)| & \preceq |\psi\{(\beta_3 + \frac{\beta_4}{2}) d(t, Q\nu)\}| \\ & \prec \{(\beta_3 + \frac{\beta_4}{2})\}|\psi d(t, Q\nu)| \\ & \prec \{(\beta_3 + \frac{\beta_4}{2})\}|d(t, Q\nu)| \end{aligned}$$

Which is a contradiction since $0 < (\beta_3 + \frac{\beta_4}{2}) < 1$ *so that our assumption* $d(t,Q\nu)$ 6= 0 *is wrong. Therefore* $t = Q\nu$*. It shows that* ν *is a coincidence point of* (Q,S)*. Thus weakly compatible of the pair* (Q,S) *yields* $QS\nu = SQ\nu$ *or* $Qt = St$*.*

Now we claim that ${}^0t^0$ *is a common fixed point of* (Q,S)*. If not then* $Qt = St$ 6= t*.*

For this we put $x = y_n$ *and* $y = t$ *in 3.9*

$$\begin{aligned} d(Py_n, Qt) &\preceq \psi\{\beta_1 d(Ry_n, St) + \beta_2 d(Py_n, Ry_n) \\ &+\beta_3 d(St, Qt) + \beta_4 \frac{d(Ry_n, Qt) + d(Py_n, St)}{2} \\ &+\beta_5 \frac{d(Ry_n, Py_n) d(Ry_n, Qt)}{1 + d(y_n, t)}\} \end{aligned}$$

$$\begin{aligned} d(t, Qt) &\preceq \psi\{\beta_1 d(t, Qt) + \beta_2 d(t, t) \\ &+\beta_3 d(Qt, Qt) + \beta_4 \frac{d(t, Qt) + d(t, Qt)}{2} \\ &+\beta_5 \frac{d(t, t) d(t, Qt)}{1 + d(y_n, t)}\} \end{aligned}$$

$$d(t, Qt) \preceq \psi\{\beta_1 d(t, Qt) + \beta_4 d(t, Qt)\}$$

$$d(t, Qt) \preceq \psi(\beta_1 + \beta_4) d(t, Qt)$$

$$\begin{aligned} |d(t, Qt)| &\preceq |\psi(\beta_1 + \beta_4) d(t, Qt)| \\ &\prec (\beta_1 + \beta_4)|\psi d(t, Qt)| \\ &\prec (\beta_1 + \beta_4)|d(t, Qt)| \end{aligned}$$

$$\implies |d(t, Qt)| \prec (\beta_1 + \beta_4)|d(t, Qt)| \qquad (3.3)$$

Which is a contradiction, since $0 < (\beta_1 + \beta_4) < 1$, *Hence our assumption that* $d(t,Qt)$ 6= 0 *is wrong.Thus* $t = Qt$. *It shows that* $t \in T(X)$ *is a common fixed point of* (Q,S).

Case II:

Similar argument arises (as in case I) if the pair (P,R) *satisfies* $(CLCS)$ *property in* $Q(X)$. *In this case* $t \in Q(X)$ *is a common fixed of* (Q,S).

Case III

Next suppose that (Q,S) *satisfies* $(CLCS)$ *- property in* $R(X)$. *Then according to definition* 2.20 *there exist a sequence* $\{y_n\}$ *in X. Such that*

$$\lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Sy_n \in R(X).$$

So there exist $t^0 = Rv^0$ *for some* $u \in X$, *where*

$$t' = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Sy_n \in R(X).$$

We claim that $Pv^0 = t^0$ *i.e.* $d(Pv^0,t^0) = 0$. *If not put* $x = v^0$ *and* $y = y_n$ *in* 3.9.

$$\begin{aligned} d(P\nu', Qy_n) &\preceq \psi\{\beta_1 d(R\nu', Sy_n) + \beta_2 d(P\nu', R\nu') \\ &+\beta_3 d(Sy_n, Qy_n) + \beta_4 \frac{d(R\nu', Qy_n) + d(P\nu', Sy_n)}{2} \\ &+\beta_5 \frac{d(R\nu', P\nu') d(Sy_n, Qy_n)}{1 + d(\nu', y_n)}\} \end{aligned} \qquad (3.4)$$

Letting $n \to \infty$

$$\begin{aligned} d(P\nu', t') &\preceq \psi\{\beta_1 d(t', t') + \beta_2 d(P\nu', t') \\ &+\beta_3 d(t', t') + \beta_4 \frac{d(t', t') + d(P\nu', t')}{2} \\ &+\beta_5 \frac{d(t', P\nu') d(t', t')}{1 + d(\nu', y_n)}\} \end{aligned} \qquad (3.5)$$

$$d(P\nu',t') \preceq \psi\{\beta_2 d(P\nu',t') + \beta_4 \frac{d(P\nu',t')}{2}\}$$

$$d(P\nu',t') \preceq \psi\{\beta_2 + \frac{\beta_4}{2}\} d(P\nu',t')$$

Which is a contradiction, since $0 < (\beta_2 + \frac{\beta_4}{2}) < 1$*, Hence our assumption that* $d(Pv^0,t^0)$ 6= 0 *is wrong.Thus* $Pv^0 = t^0$*. It shows that* $t^0 \in R(x)$ *is a coincidence point of* (Q,S)*. Thus weakly compatible of the pair* (P,R) *yields* $PRv^0 = RPv^0 = Pt^0 = Rt^0$*.*

Case(IV): Now we claim that t^0 *is a common fixed point of* (P,R)*. If not then* $Pt^0 = Rt^0$ 6= t*.*

For this we put $x = t^0$ *and* $y = y_n$ *in 3.9*

$$\begin{aligned} d(Pt',Qy_n) \preceq & \ \psi\{\beta_1 d(Rt',Sy_n) + \beta_2 d(Pt',Rt') \\ & + \beta_3 d(Sy_n,Qy_n) + \beta_4 \frac{d(Rt',Qy_n) + d(Pt',Sy_n)}{2} \\ & + \beta_5 \frac{d(Rt',Pt')d(Sy_n,Qy_n)}{1 + d(t',y_n)}\} \end{aligned}$$

$n \to \infty$

$$d(Pt',t') \preceq \psi\{\beta_1 d(Pt',t') + \beta_4 \frac{d(Pt',t') + d(Pt',t')}{2}\}$$

$$\begin{aligned} d(Pt',t') &\preceq \psi\{\beta_1 + \beta_4\} d(Pt',t') \\ |d(Pt',t')| &\preceq |\psi\{\beta_1 + \beta_4\} d(Pt',t')| \\ &\prec \{\beta_1 + \beta_4\} |\psi(Pt',t')| \\ &\prec \{\beta_1 + \beta_4\} |d(Pt',t')| \end{aligned}$$

Letting $\Longrightarrow |d(Pt',t')| \prec \{\beta_1 + \beta_4\} |d(Pt',t')|$

Which is a contradiction, since $0 < (\beta_1 + \beta_4) < 1$*, Hence our assumption that* $d(Pt^0,t^0)$ 6= 0 *is wrong.Thus* $Pt^0 = t^0$*. It shows that* $t \in R(X)$ *is a common fixed point of* (P,R)*.*

Case V:

Similar argument arises if the pair (Q,S) *satisfy* $(CLCS)$*-property in* $P(X)$*. In this case* $t^0 \in P(X)$ *is a common fixed point of* (P,R)*.*

Further, we claim that the common fixed point t^0 *of* (P,R) *and* t *of* (Q,S) *are same, i.e..* $t = t^0$*. If not, then put* $x = t$ *and* $y = t^0$ *in* 3.9*, we have*

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$$\begin{aligned} d(Pt', Qt) &\preceq \psi\{\beta_1 d(Rt', St) + \beta_2 d(Pt', Rt') \\ &+\beta_3 d(St, Qt) + \beta_4 \frac{d(Rt', Qt) + d(Pt', St)}{2} \\ &+\beta_5 \frac{d(Rt', Pt')d(St, Qt)}{1 + d(t, t')}\} \end{aligned} \quad (3.6)$$

Letting $n \to \infty$

$$\begin{aligned} d(t', t) &\preceq \psi\{\beta_1 d(t', t) + \beta_2 d(t', t') \\ &+\beta_3 d(t, t) + \beta_4 \frac{d(t', t) + d(t', t)}{2} \\ &+\beta_5 \frac{d(t', t')d(t, t)}{1 + d(t, t')}\} \end{aligned} \quad (3.7)$$

$$d(t', t) \preceq \psi\{\beta_1 d(t', t) + \beta_4 d(t', t)\}$$

$$\begin{aligned} \implies |d(t', t)| &\preceq |\psi\{\beta_1 d(t', t) + \beta_4 d(t', t)\}| \\ |d(t', t)| &\prec \{\beta_1 + \beta_4\}|\psi d(t', t)| \\ |d(t', t)| &\prec \{\beta_1 + \beta_4\}|d(t', t)| \end{aligned} \quad (3.8)$$

Equation 3.8 is again a contradiction. Thus the assumption of $d(t,t^0)$ 6= 0 *is wrong. So* $t = t^0$*. It shows that t is common fixed point of all the four mappings P,Q,R and S.*
The uniqueness of the common fixed point of P,Q,R and S. is easy consequence of the common point of coincidence of the pair (P,R) *and* (Q,S)*. Also, proof is similar in case* $u \in P(X)$*. This completes the proof.*

On taking $\beta_2 = \beta_3 = \beta_4 = 0$ and $\psi = R = S = I$, we have the following corollary which is a result proved by [1].

Corollary 3.2. *Let* (X,d) *be a complex valued metric space. Suppose that the mapping P and Q be self maps on X satisfying the following condition*

$$d(Px, Qy) \preceq \beta_1 d(x, y) + \beta_5 \frac{d(x, Px)d(y, Qy)}{1 + d(x, y)}$$

Where $\beta_1 + \beta_5 < 1$ *then the mappings P and Q have a common fixed point in X.*

Theorem 3.3. *Let* (X,d) *be a complex valued metric space. Suppose that the mapping P,Q,R and S be four self maps on X satisfying the following condition*

(1) (P,R) *satisfy* $(CLCS)$ *- property in* $S(X) \cap Q(X)$ *and* (Q,S) *satisfy* $(CLCS)$ *- property in* $P(X) \cap R(X)$

(2)

$$\begin{aligned} d(Px, Qy) &\preceq \psi\{\beta_1 d(Rx, Sy) + \beta_2 d(Px, Rx) \\ &+\beta_3 d(Sy, Qy) + \beta_4 \frac{d(Rx, Qy) + d(Px, Sy)}{2} \\ &+\beta_5 \frac{d(Rx, Px)d(Sy, Qy)}{1 + d(x, y)}\} \end{aligned} \quad (3.9)$$

∀ *x,y* ∈ *X,where* $\sum_{i=1}^{i=5} \beta_i < 1$ *and all* $\beta_i > 0$. *If* (*P,R*) *and* (*Q,S*) *are weakly compatible then mappings P,Q,R and S have a unique common fixed point in X.*

Proof:

For the proof of section (1) of theorem,

(1) *We combined Case*(*I*) *and Case*(*II*) *to show* (*P,R*) *satisfy* (*CLCS*) *property in* $S(X) \cap Q(X)$ *3.1.*

(2) *We combined Case*(*III*) *and Case*(*IV*) *to show* (*Q,S*) *satisfy* (*CLCS*) *property in* $P(X) \cap R(X)$ *3.1.*

The proof of section (2) of theorem runs exactly same as theorem 3.1.

4. Conclusion

Results obtained in this research paper are totally new with defining (*CLCS*) - property on four different type mappings. Contraction used in the paper in theorems are totally new. This paper opens new challenges to the fixed point writers.

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