TOTAL OUTER INDEPENDENT GEODETIC DOMINATION NUMBER OF A GRAPH

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Abstract. In this paper the concept of the *total outer independent geodetic domination number* of a graph G is introduced. An outer independent geodetic dominating set $S \subseteq V(G)$ is said to be a total outer independent geodetic dominating set of G if the subgraph $< S >$ has no isolated vertices. The minimum cardinality of a total outer independent geodetic dominating set is called the total outer independent geodetic domination number and is denoted by $\gamma_{gt}^{oi}(G)$. Some general properties satisfied by this concept are studied. The total outer *independent geodetic domination number* of certain classes of graphs are determined. It is shown that for every pair m, n of integers with $3 \leq m \leq n$, there exist a connected graph G of order n such that $\gamma_{gt}^{oi}(G) = m$. Also, it is shown that for any three integers p, q and r such that $2 \leq p \leq q \leq r$ there exists a connected graph G with $g(G) = p$, $\gamma_g(G) = q$ and $\gamma_{gt}^{oi}(G) = r$.

Key Words: independent geodetic domination number, outer independent geodetic domination number.

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Introduction

By a graph $G = (V, E)$, we mean a simple graph of order at least two. The order and size of G are denoted by p and q respectively. For basic theoretic terminology, (see [1]). The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup N\{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex (see [2]).

For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A u - v path of length $d(u,v)$ is called a $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets(see [3]).

A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbor in it. The size of a minimum dominating set in a graph G is called the domination number of G

and is denoted by $\gamma(G)$. A geodetic dominating set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic dominating set is denoted by $\gamma_g(G)$ (see [4],[5],[6],[7]). A geodetic dominating set $S \subseteq V(G)$ in a graph G is said to be a total geodetic dominating set if $< S >$ has no isolated vertices (see [8,9]).

An independent set $S \subseteq V(G)$ in a graph G such that no two vertices in S are adjacent in G. The maximum cardinality of an independent set is called the independence number and is denoted by $\alpha(G)$. A geodetic dominating set S in a graph G is said to be an independent geodetic dominating set if $< S >$ is independent. The minimum cardinality of an independent geodetic dominating set is called the independent geodetic domination number and is denoted by $\gamma_g^{\,i}(G)$. A geodetic dominating set S in a graph G is said to be an outer independent geodetic dominating set if $< V - S >$ is independent. The minimum cardinality of an outer independent geodetic dominating set is called the outer independent geodetic domination number and is denoted by $\gamma_g^{\,oi}(G)$ (see [10]).

1. Total Outer Independent geodetic Domination Number of a graph

Definition 1.1. An outer independent geodetic dominating set $S \subseteq V$ is said to be a total outer *independent geodetic dominating set* of a graph G the subgraph $< S >$ has no isolated vertices. The minimum cardinality of a total outer independent geodetic dominating set is called the total outer *independent geodetic domination number of* G and is denoted by $\gamma_{gt}^{\,oi}(G)$.

Example 1.2. For the graph G given in Figure 1.1, $S = \{v_1, v_2, v_3, v_5, v_6, v_8, v_9\}$ is a total outer independent geodetic dominating set of G and is minimum so that $\gamma_{gt}^{\,oi}(G) = 7$.

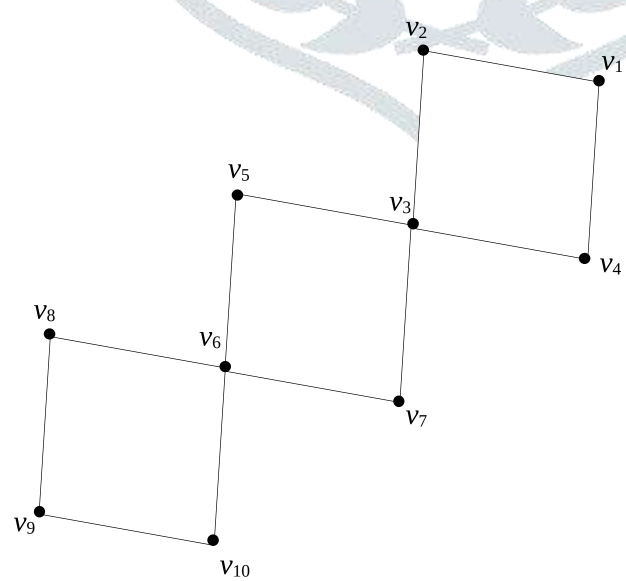

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Figure 1.1

Remark 1.3. For the graph given in Figure 1.1, $S = \{v_1, v_3, v_6, v_9\}$ is an outer independent geodetic dominating set of G and is minimum so that $\gamma_g^{oi}(G) = 4$. Thus the outer independent geodetic domination number and the total outer independent geodetic domination number of a graph are different.

Remark 1.4. There can be more than one total outer independent geodetic dominating set for a graph. For the graph given in Figure 1.1, $S_1 = \{v_1, v_2, v_3, v_5, v_6, v_8, v_9\}$ and $S_2 = \{v_1, v_3, v_4, v_6, v_7, v_9, v_{10}\}$ are two total outer independent geodetic dominating sets.

Theorem 1.5. For any connected graph G of order p, $2 \le \gamma_g^{oi}(G) \le \gamma_{gt}^{oi}(G) \le p$.

Proof. An independent geodetic dominating set needs at least two vertices and therefore $\gamma_g^{oi}(G) \ge 2$. Since every total outer independent geodetic dominating set is an outer independent geodetic dominating set, $\gamma_g^{oi}(G) \le \gamma_{gt}^{oi}(G)$. Since the number of vertices of G is p, $2 \le \gamma_g^{i}(G) \le \gamma_{gt}^{oi}(G) \le p$.

Remark 1.6. The bounds in Theorem 1.5 are sharp. For the graph given in Figure 1.1, $\gamma_g^{oi}(G) = 4$, $\gamma_{gt}^{oi}(G) = 7$ and $p = 10$.

Corollary 1.7. For the star graph $K_{1,n}$ $(n \ge 1)$, $\gamma_{gt}^{oi}(K_{1,n}) = n + 1$.

Theorem 1.8. For every integer $n \ge 2$ we have

$$\gamma_{gt}^{oi}(P_n) = \begin{cases} 3 + (\frac{n-2}{2}) & if\ n\ is\ even \\ \left\lceil \frac{n+1}{3} \right\rceil + 3 & if\ n\ is\ odd \end{cases}$$

Theorem 1.9. Let $W_n = K_1 + C_{n-1}$ is a wheel, $n \ge 5$, then

$$\gamma_{gt}^{oi}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Theorem 1.10. For every integer $m, n \ge 1$ we have

$$\gamma_{gt}^{oi}(K_{m,n}) = \begin{cases} 2, & m = n = 1 \\ m + n, & m = 1\ and\ n \ge 2 \\ \min\{m, n\} + 1, & m, n \ge 2 \end{cases}$$

Observation 1.11. For every disjoint graphs $G_1, G_2, \ldots, G_k$ we have $\gamma_{gt}^{oi}(G_1 \cup G_2 \cup \ldots \cup G_k) = \gamma_{gt}^{oi}(G_1) + \gamma_{gt}^{oi}(G_2) + \ldots + \gamma_{gt}^{oi}(G_k)$

Observation 1.12. For every graph G of order n we have $\gamma_{gt}^{oi}(G) \ge n - \alpha(G)$.

Theorem 1.13. For every graph G we have $\delta(G) \le \gamma_{gt}^{oi}(G) \le \Delta(G)$.

Remark 1.14. For any connected graph G, $\gamma_{gt}(G) \le \gamma_{gt}^{oi}(G)$.

Theorem 1.15. For any pair of positive integers m and n such that $9 < m \leq n$, there is a connected graph G of order n and $\gamma_{gt}^{oi}(G) = m$.

Proof.

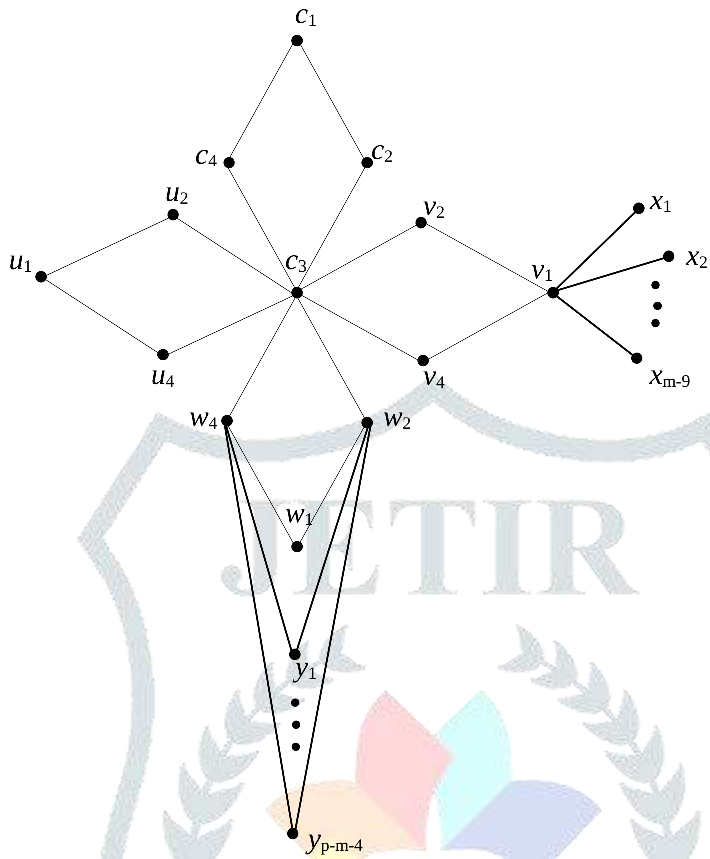

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Figure 1.2

Let C_1:c_1,c_2,c_3,c_4 ; C_2: u_1,u_2,u_3,u_4; C_3: v_1,v_2,v_3,v_4 and C_4: w_1,w_2,w_3,w_4 be four cycles on the corresponding four vertices. Let H_1 be a graph by identifying the three vertices u_3, v_3 and w_3 in the cycles C_2, C_3 and C_4 with the vertex c_3 in C_1. Now, let H_2 be a graph by adding a set of non-adjacent vertices $x_1,x_2,\ldots,x_{m-9}$ and joining each vertex $x_i(1 \leq i \leq m\text{-}9)$ with the vertex v_1. Let G be graph by joining a set of vertices $y_1,y_2,\ldots,y_{p-m-4}$ with both the vertices w_2 and w_4 as shown in the Figure 1.2.

Let $S_1 = \{c_1, c_3, u_1, v_1, w_2, w_4, x_1, x_2, \ldots, x_{m-9}\}$is an outer independent geodetic dominating set of G but it is not a total outer independent geodetic dominating set of G. Hence $S_2 = S_1 \cup \{c_2, w_1, u_2\}$ is a minimum total outer independent geodetic dominating set of G so that $\gamma_{gt}^{oi}(G) = m$.

Theorem 1.16. For any three integers p, q, r such that $2 \leq p \leq q \leq r$, there exists a connected graph G with $g(G) = p$, $\gamma_g(G) = q$ and $\gamma_{gt}^{oi}(G) = r$.

Proof.

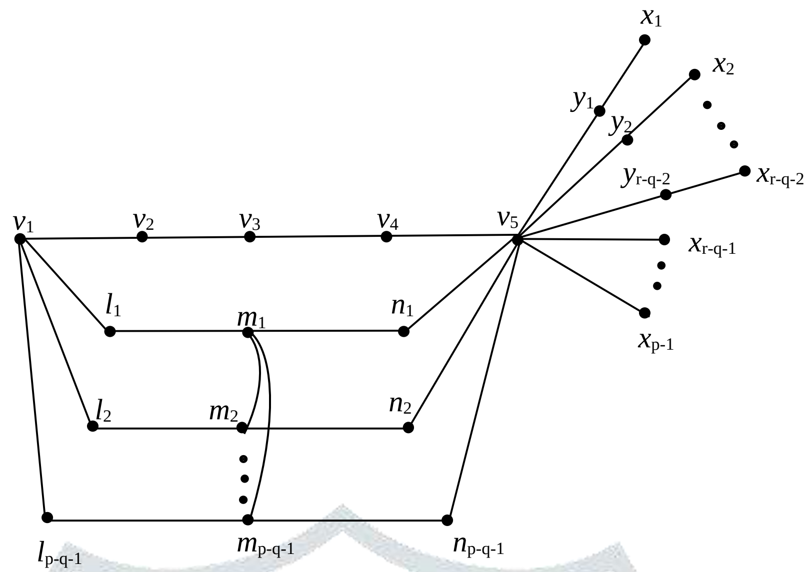

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Figure 1.3

Let P_5: v_1, v_2, v_3, v_4, v_5 be a path on five vertices. Let P_i : i_i, m_i, n_i, $(1 \leq i \leq p - q - 1)$ be paths on three vertices. Let H_1 be a graph obtained from P and P_i by joining each li $(1 \leq i \leq p - q - 1)$ with the vertex v_1 and joining each vertex $n_i(1 \leq i \leq p - q - 1)$ with the vertex v_5. Let H_2 be a graph by Taking a copy of star $K_{1,p-1}$ with leaves $x_1, x_2, \ldots, x_{p-1}$ and the support vertex v_5. Subdivide the edges xx_i , where $1 \leq i \leq r - q - 2$,calling the new vertices $y_1, y_2, \ldots, y_{r-q-2}$ where x_i is adjacent to y_i, and y_i is adjacent to v_5 for all $i \in \{1,2,\ldots,r - q - 2\}$. Let G be the given graph as shown in Figure 1.3.

Let $S_1 = \{v_1, x_1, x_2, \ldots, x_{p-1}\}$. Then S_1 is a minimum geodetic set of G. Then $g(G) = p$. Now, let $S_2 = S_1 \cup \{v_4, m_1, m_2, \ldots, m_{q-p-1}\}$. Clearly S_2 is a minimum geodetic dominating set of G which contains q vertices so that $\gamma_g(G) = q$. Also, let $S_3 = S_2 \cup \{v_2, v_5, y_1, y_2, \ldots, y_{r-q-2}\}$. Clearly, S_3 is a minimum total outer independent geodetic dominating set of G which contains r vertices so that $\gamma_{gt}^{oi}(G) = r$.

Theorem 1.17. For every pair of integers a, b such that $2 \leq a \leq b$, there exists a connected graph G with $g(G) = a$ and $\gamma_{gt}^{oi}(G) = b$.

Proof.

Case(i). Let $a = b$.

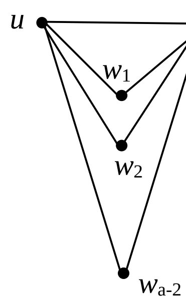

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Figure 1.4

Let P_2: u, v be a path on two vertices. Let G be a graph by adding a set of vertices w_1, w_2,…,w_{a-2} and joining each $w_i(1 \leq i \leq a - 2)$ with the two vertices u and v as shown in Figure 1.4.

Let $S = \{u, v, w_1, w_2,…,w_{a-2}\}$. Then Clearly, S is both the minimum geodetic and minimum total outer independent geodetic dominating set of G. Therefore $g(G) = \gamma_{gt}^{oi}(G) = a = b$.

Case (ii). Let $a + 1 = b$.

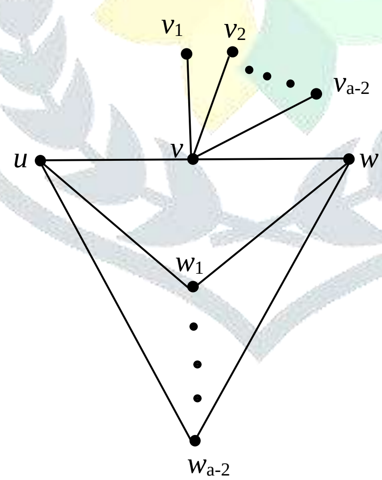

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Figure 1.5

Let P_3: u, v, w be a path on three vertices. Let H be a graph by obtaining from P_3 by adding a set of new vertices v_1, v_2,…,v_{a-2} and joining each vertex v_i $(1 \leq i \leq a – 2)$ with the vertex v. Let G be a graph by adding a set of vertices w_1, w_2,…,w_{a-2} and joining each $w_i(1 \leq i \leq a - 2)$ with the two vertices u and w as shown in Figure 1.5.

Let $S_1 = \{u, w, v_1, v_2,\ldots,v_{a-2}\}$. Clearly, S is a minimum geodetic set of G so that $g(G) = a$. Let $S_2 = S_1 \cup \{v\}$ is a minimum total outer independent geodetic dominating set of G. Therefore $\gamma_{gt}^{oi}(G) = a + 1$.

Case (iii). Let $a + 2 = b$.

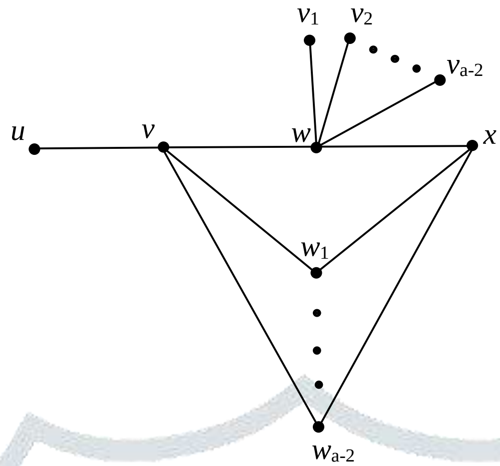

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Figure 1.6

Let P_4: u, v, w, x be a path on four vertices. Let H be a graph by obtaining from P_4 by adding a set of new vertices $v_1, v_2,\ldots,v_{a-2}$ and joining each vertex v_i $(1 \leq i \leq a - 2)$ with the vertex w. Let G be a graph by adding a set of vertices $w_1, w_2,\ldots,w_{a-2}$ and joining each $w_i(1 \leq i \leq a - 2)$ with the two vertices v and x as shown in Figure 1.6.

Let $S_1 = \{u, x, v_1, v_2,\ldots,v_{a-2}\}$. Clearly, S is a minimum geodetic set of G so that $g(G) = a$. Let $S_2 = S_1 \cup \{v, w\}$ is a minimum total outer independent geodetic dominating set of G. Therefore $\gamma_{gt}^{oi}(G) = a + 2$.

Case (iv). Let $a + 3 \leq b$.

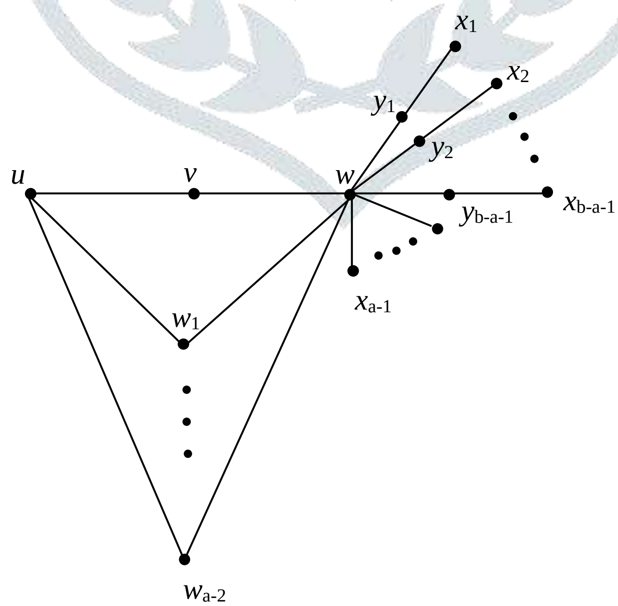

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Figure 1.7

Let P_3: u, v, w be a path on three vertices. Let H_1 be a graph by obtained from P_3 by adding a set of new vertices $w_1, w_2,\ldots,w_{a-2}$ and joining each vertex w_i $(1 \leq i \leq a - 1)$ with the vertices u and w. Let H_2 be a graph by taking a copy of star $K_{1,a-1}$ with leaves $x_1, x_2,\ldots,x_{a-1}$ and the support vertex w. Subdivide the edges xx_i, where $1 \leq i \leq b - a - 1$,calling the new vertices $y_1,y_2,\ldots,y_{b-a-1}$ where x_i is adjacent to y_i, and y_i is adjacent to w for all $i \in \{1,2,\ldots,b - a - 1\}$. Let G be the given graph as shown in Figure 1.7.

Let $S_1 = \{u, x_1, x_2,\ldots,x_{a-1}\}$. Clearly, S is a minimum geodetic set of G so that $g(G) = a$. Let $S_2 = S_1 \cup \{w, y_1, y_2,\ldots,y_{b-a-1}\}$ is a minimum total outer independent geodetic dominating set of G and so $\gamma_{gt}^{oi}(G) = b$.

REFERENCES

[1] Buckley and F.Harary. Distance in Graphs, Addision-Wesley, Redwood City, (1990).

[2] Buckley, F.Harary and L.V.Quintas, Extremal results on the Geodetic Number of a Graph, Sci.Ser. A, Vol.2, 17 – 26.

[3] G. Chartrand, F.Harary, P. Zhang, On the Geodetic Number of a Graph, Networks, Vol. 39 (2002), 1- 6.

[4] H. Escuadro, R. Gera, A. Hansberg, N. Jafari Rad, L. Voltmann, Geodetic domination in graphs, J. Combin. Math. Combin. Comput., Vol. 77 (2011), 89 – 101.

[5] Gravier S and Mollard M, On domination numbers of Cartesian product of paths, *Discrete Mathematics*, 80 (1997) 247–250.

[6] A.Hansberg. L. Volkmann. On The Geodetic and Geodetic Domination Numbers of a Graph. Discrete Mathematics, Vol. 310(15), 2140 – 2146 (28 August 2010).

[7] T.W. Haynes, S.T Hedetniemi and P.J Slater, Fundamentals of Domination in Graphs, Vol. 208, Marcel Dekker Inc., New York, 1998.

[8] Maryan Atapour and Nasrin Soltankhah, On Total Domination in Graphs, Int J.Contemp. Math. Sciences, Vol 4(253-257), 2009.

[9] Michael A. Henning, A Survey of Selected Recent Results on Total Domination in Graphs, Discrete Mathematics Vol. 309 (32 – 63) (6 Jan 2009).

[10] VenkanaGouda M Goudar, Total Outer Independent Geodetic Number of a Graph, Journal of Ultra Scientist of Physical Sciences-A, 30(4), (2018) 247-256.