L(3,1) - LABELING FOR SOME EXTENDED DUPLICATE GRAPHS

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Abstract: In this paper, we investigate the extended duplicate graph of triangular snake graph and quadrilateral snake graph admits L(3,1) - labeling.

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1. Introduction

Graph labeling is one of the most important area in graph theory. The concept of graph labeling was introduced by Rosa in 1967 [4]. For a dynamic survey of various graph labeling we refer to J.A.Gallian[3]. Griggs and Yen[5] defined L(2,1) labeling of the graph G=(V,E) where f is a function which assigns labels to every u,v∈V from the set of positive integer such that |f(u) – f(v)| ≥ 2 if d(u,v) = 1 and |f(u) – f(v) | ≥ 1 if d(u,v) = 2.

E.Sampthkumar [1] introduced the concept of duplicate graph. Thirusangu et al.[2] have introduced the concept of extended duplicate graph. G.J.Chang and D.Kuo et al., on L(d,1)-Labeling of graphs. In L(3,1)-labeling of a graph G=(V,E) where f is a function which assigns label to every u,v∈V from the set of positive integer such that |f(u)-f(v)| ≥3 if d(u,v) = 1 and |f(u) – f(v) | ≥ 1 if d(u,v) = 2. The L(3,1) labeling number, λ(G) of G is the smallest number λ such that G has an L(3,1) labeling with λ as the maximum label.

2. Preliminaries

In this section, we give the basic definitions which are relevant to this paper. Let G(V,E) be a finite, simple and undirected graph with p vertices and q edges.

Definition:2.1 Triangular snake

A triangular snake TSₘ is obtained from a path u₁, u₂, …, uₘ₊₁ by connecting uᵢ and uᵢ₊₁ to a new vertex vᵢ, for 1 ≤ i ≤ m, where ‘m’ is the number of edges of the path.
**Definition: 2.2 Quadrilateral snake graph**

A quadrilateral snake $QS_m$ is obtained from a path $u_1,u_2,u_3, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertex $v_i$ and $w_i$ respectively and then joining $v_i$ and $w_i$, $1 \leq i \leq n-1$, where ‘m’ is the number of edges of the path. In general, a quadrilateral snake has $3m+1$ vertices and $4m$ edges.

**Definition: 2.3 Duplicate graph**

A Simple graph G with vertex set $V$ and edge set $E$. The duplicate graph of G is $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \emptyset$ and $h : V \rightarrow V'$ is bijective. The edge set $E_1$ of DG is defined as the edge $ab \in E$ iff both edges $ab'$ and $a'b$ are in $E_1$.

**Definition: 2.4 Extended duplicate graph of triangular snake**

Let $DG = (V_1,E_1)$ be a duplicate graph of the triangular snake graph $G(V,E)$. Extended duplicate graph of triangular snake is obtained by adding the edge $v_2v'_2$ to the duplicate graph and it is denoted by $EDG(TS_m)$. Clearly it has $4m+2$ vertices and $6m+1$ edges, where ‘m’ is the number of edges.

**Definition: 2.5 Extended duplicate graph of quadrilateral snake**

Let $DG = (V_1,E_1)$ be a duplicate graph of the quadrilateral snake graph $G(V,E)$. Extended duplicate graph of quadrilateral snake graph is obtained by adding the edge $v_2v'_2$ to the duplicate graph and it is denoted by $EDG(QS_m)$. Clearly it has $6m+2$ vertices and $8m+1$ edges, where ‘m’ is the number of edges.
Definition : 2.6 \( L(2,1) \) – Labeling

An \( L(2,1) \) labeling or distance two labeling of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of all non-negative integers such that \( |f(x) - f(y)| \geq 2 \) if \( d(x, y) = 1 \) and \( |f(x) - f(y)| \geq 1 \) if \( d(x, y) = 2 \). The \( L(2,1) \) labeling number \( \lambda(G) \) of \( G \) is the smallest number \( k \) such that \( G \) has an \( L(2,1) \) labeling with max\{\( f(v), v \in V(G) \)\} = \( k \).

Definition : 2.7 \( L(3,1) \) – Labeling

Let \( G \) be a graph with set of vertices \( V \) and set of edges \( E \). Let \( f \) be a function \( f: V \rightarrow \mathbb{N} \), where \( f \) is an \( L(3,1) \)-labeling of \( G \) if, for all \( u, v \in V \), \( |f(u) - f(v)| \geq 3 \) if \( d(u,v) = 1 \) and \( |f(u) - f(v)| \geq 1 \) if \( d(u,v) = 2 \).

Definition : 2.8

The difference between maximum and minimum values of \( f \) for all possible \( f \) is called span of the labeling, and it is denoted by \( \lambda_{3,1}(G) \) or \( \lambda(G) \) or \( \lambda \), positive integer \( \lambda \) to be used to label a graph \( G \) by \( L(3,1) \)-labeling.

3. MAIN RESULTS

3.1: \( L(3,1) \)-LABELING FOR TRIANGULAR SNAKE GRAPH \( EDG(TS_m) \), \( m \geq 1 \)

Here, we present an algorithm and prove the existence of \( L(3,1) \)-labeling for \( EDG(TS_m) \).

Algorithm-1

Procedure - [\( L(3,1) \)-labeling for \( EDG(TS_m) \), \( m \geq 1 \)]

\( V \leftarrow \{ v_1, v_2, v_3, \ldots, v_{2m}, v_{2m+1}, v'_1, v'_2, v'_3, \ldots, v'_{2m}, v'_{2m+1} \} \)

\( E \leftarrow \{ e_1, e_2, e_3, \ldots, e_{3m}, e_{3m+1}, e'_1, e'_2, \ldots, e'_{3m} \} \)

\( v_1 \leftarrow 0, v_2 \leftarrow 7, v_3 \leftarrow 6, v_4 \leftarrow 1, v_5 \leftarrow 8 \)

\( v'_1 \leftarrow 0, v'_2 \leftarrow 3, v'_3 \leftarrow 4, v'_4 \leftarrow 1, v'_5 \leftarrow 9 \)

for \( i = 0 \) to \((m-3)/4\) do

\( v_{6+8i} \leftarrow 0; v_{7+8i} \leftarrow 2; v'_{6+8i} \leftarrow 5; v'_{6+8i} \leftarrow 3 \)

end for

for \( i=0 \) to \((m-4)/4\) do

\( v_{8+8i} \leftarrow 6; v_{9+8i} \leftarrow 9; v'_{8+8i} \leftarrow 6; v'_{9+8i} \leftarrow 10 \)

end for

for \( i=0 \) to \((m-5)/4\) do

\( v_{10+8i} \leftarrow 0; v_{11+8i} \leftarrow 1; v'_{10+8i} \leftarrow 5; v'_{11+8i} \leftarrow 4 \)

end for
for i = 0 to (m-6)/4 do
  \( v_{12+8i} \leftarrow 7; v_{13+8i} \leftarrow 10; v'_{12+8i} \leftarrow 6; v'_{13+8i} \leftarrow 11 \)
end for
end procedure

**Theorem 3.1**: The extended duplicate graph of triangular snake graph admits \( L(3,1) \)-labeling and its number \( \lambda(G) \) is 11.

**Proof**: Let \( TS_m \) be the triangular snake graph and \( EDG(TS_m) \) be the extended duplicate graph of triangular snake graph.

Define the set of vertices and edges are

\[
V(G) = \{v_1, v_2, v_3, \ldots, v_{2m}, v'_{1}, v'_{2}, v'_{3}, \ldots, v'_{2m}, v'_{2m+1}\}
\]

\[
E(G) = \{e_1, e_2, e_3, \ldots, e_{3m}, e_{3m+1}, e'_{1}, e'_{2}, \ldots, e'_{3m}\}
\]

Let \( V(G) = V_1(G) \cup V_2(G) \),

Where \( V_1 = \{ v_i / 1 \leq i \leq 2m+1 \} \)

\( V_2 = \{ v'_i / 1 \leq i \leq 2m+1 \} \)

For \( V_1 \) and \( V_2 \), we define a mapping \( f: V(G) \rightarrow \mathbb{N} \cup \{0\} \) such that \( |f(x) - f(y)| \geq 3 \) if \( d(x,y) = 1 \) and \( |f(x) - f(y)| \geq 1 \) if \( d(x,y) = 2 \).

Using algorithm 1, the vertices \( v_1, v_2, v_3, v_4, v_5, v'_{1}, v'_{2}, v'_{3}, v'_{4} \) and \( v'_{5} \) receive the labels 0, 7, 6, 1, 8, 0, 3, 4, 1 and 9 respectively;

i) \( f(v_{6+8i}) = 0 \) and \( f(v'_{6+8i}) = 5 \) for \( 1 \leq i \leq (m-3)/4 \)

ii) \( f(v_{7+8i}) = 2 \) and \( f(v'_{7+8i}) = 3 \) for \( 1 \leq i \leq (m-3)/4 \)

iii) \( f(v_{8+8i}) = 6 \) and \( f(v'_{8+8i}) = 6 \) for \( 1 \leq i \leq (m-4)/4 \)

iv) \( f(v_{9+8i}) = 9 \) and \( f(v'_{9+8i}) = 10 \) for \( 1 \leq i \leq (m-4)/4 \)

v) \( f(v_{10+8i}) = 0 \) and \( f(v'_{10+8i}) = 5 \) for \( 1 \leq i \leq (m-5)/4 \)

vi) \( f(v_{11+8i}) = 1 \) and \( f(v'_{11+8i}) = 4 \) for \( 1 \leq i \leq (m-5)/4 \)

vii) \( f(v_{12+8i}) = 7 \) and \( f(v'_{12+8i}) = 6 \) for \( 1 \leq i \leq (m-6)/4 \)

viii) \( f(v_{13+8i}) = 10 \) and \( f(v'_{13+8i}) = 11 \) for \( 1 \leq i \leq (m-6)/4 \)

Thus all the vertices are labeled.

Now to prove that \( L(3,1) \)-labeling number \( \lambda(G) \) is 11.

**Case 1**: Let \( x, y \) be any two vertices in \( V_1(G) \).

Subcase (i): For \( m = 1 \)

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{m+1} \), \( y = v_{m+2} \) then \( f(x) = 7 \) and \( f(y) = 6 \), \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x)-f(y)| = 2+|7-6| = 2+1 = 3 \geq 3 \).
Subcase (ii): For \( m = 2 \)

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{m+2} \), \( y = v_{m+3} \) then \( f(x) = 1 \) and \( f(y) = 8 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 7 = 9 \geq 3 \).

Subcase (iii): For \( m = 4n - 1 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{6+8i} \), \( y = v_{7+8i} \) then \( f(x) = 0 \) and \( f(y) = 2 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 2 = 4 \geq 3 \).

Subcase (iv): For \( m = 4n \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{6+8i} \), \( y = v_{7+8i} \) then \( f(x) = 0 \) and \( f(y) = 2 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 2 = 3 \geq 3 \).

Subcase (v): For \( m = 4n + 1 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{8+8i} \), \( y = v_{9+8i} \) then \( f(x) = 6 \) and \( f(y) = 1 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 3 = 5 \geq 3 \).

Subcase (vi): For \( m = 4n + 2 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \) such that \( x = v_{10+8i} \), \( y = v_{11+8i} \) then \( f(x) = 7 \) and \( f(y) = 10 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 3 = 5 \geq 3 \).

Case 2: Let \( x \) and \( y \) be any two vertices in \( V_2(G) \).

Subcase (i): For \( m = 1 \)

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{m+1} \), \( y = v'_{m+2} \) then \( f(x) = 3 \) and \( f(y) = 4 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 1 = 3 \).

Subcase (ii): For \( m = 2 \)

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{m+2} \), \( y = v'_{m+3} \) then \( f(x) = 1 \) and \( f(y) = 9 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 8 = 10 \geq 3 \).

Subcase (iii): For \( m = 4n - 1 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{6+8i} \), \( y = v'_{7+8i} \) then \( f(x) = 5 \) and \( f(y) = 3 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 2 = 4 \geq 3 \).

Subcase (iv): For \( m = 4n \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{8+8i} \), \( y = v'_{9+8i} \) then \( f(x) = 6 \) and \( f(y) = 10 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 4 = 6 \geq 3 \).

Subcase (v): For \( m = 4n + 1 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{10+8i} \), \( y = v'_{11+8i} \) then \( f(x) = 5 \) and \( f(y) = 4 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 1 = 3 \).

Subcase (vi): For \( m = 4n + 2 \), \( n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \) such that \( x = v'_{12+8i} \), \( y = v'_{13+8i} \) then \( f(x) = 6 \) and \( f(y) = 11 \), \( d(x, y) = 2 \). Therefore \( d(x, y) + |f(x) - f(y)| = 2 + 5 = 7 \geq 3 \).
Case 3: Let $x$ and $y$ be any two vertices in $V_1(G)$ and $V_2(G)$.

Subcase (i): For $m=1$

Let $x$ and $y$ be any two adjacent vertices on $V_1(G)$ & $V_2(G)$ such that $x = v_{m+1}$, $y = v'_{m+1}$ then $f(x) = 7$ and $f(y) = 3$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |7 - 3| = 1 + 4 = 5 \geq 3$.

Subcase (ii): For $m=2$,

Let $x$ and $y$ be any two adjacent vertices on $V_1(G)$ and $V_2(G)$ such that $x = v_{m+2}$, $y = v'_{m+3}$ then $f(x) = 1$ and $f(y) = 9$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |1 - 9| = 1 + 8 = 9 \geq 3$.

Subcase (iii): For $m=4n-1$, $n \in \mathbb{N}$.

Let $x$ and $y$ be any two adjacent vertices on $V_1(G)$ & $V_2(G)$ such that $x = v_{6i+8i}$, $y = v'_{7i+8i}$ then $f(x) = 0$ and $f(y) = 3$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |0 - 3| = 1 + 3 = 4 \geq 3$.

Subcase (iv): For $m=4n$, $n \in \mathbb{N}$.

Let $x$ and $y$ be any two adjacent vertices on $V_1(G)$ & $V_2(G)$ such that $x = v_{8i+8i}$, $y = v'_{9i+8i}$ then $f(x) = 6$ and $f(y) = 10$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |6 - 10| = 1 + 4 = 5 \geq 3$.

Subcase (v): For $m=4n+1$, $n \in \mathbb{N}$.

Let $x$ and $y$ be any two non-adjacent vertices on $V_1(G)$ & $V_2(G)$ such that $x = v_{10i+8i}$, $y = v'_{11i+8i}$ then $f(x) = 0$ and $f(y) = 4$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |0 - 4| = 1 + 4 = 5 \geq 3$.

Subcase (vi): For $m=4n+2$, $n \in \mathbb{N}$.

Let $x$ and $y$ be any two adjacent vertices on $V_1(G)$ & $V_2(G)$ such that $x = v_{12i+8i}$, $y = v'_{13i+8i}$ then $f(x) = 7$ and $f(y) = 11$, $d(x,y) = 1$. Therefore $d(x,y) + |f(x) - f(y)| = 1 + |7 - 11| = 1 + 4 = 5 \geq 3$.

Thus, by continuing this process of $x$ and $y$, we get

$$d(x,y) + |f(x) - f(y)| = \begin{cases} 
\geq 3 & \text{if } d = 2 \\
\geq 4 & \text{if } d = 1
\end{cases}.$$

Hence, the extended duplicate graph of triangular snake graph admits $L(3,1)$-labeling and its number $\lambda(G)$ is 11.
Example 1: L(3,1)-labeling diagram in EDG(TS₅) and EDG(TS₆) is shown in figures (1) & (2)

3.2: L(3,1)-LABELING FOR QUADRILATERAL SNAKE GRAPH EDG(QSₘ), m ≥ 1

Here, we present an algorithm and prove the existence of L(3,1)-labeling for EDG(QSₘ).
Algorithm 2

Procedure – \([L(3,1)]\)-labeling for EDG(QSm), \(m \geq 1\)

\[ V \leftarrow \{ v_1, v_2, v_3, \ldots, v_{3m}, v_{3m+1}, v'_1, v'_2, v'_3, \ldots, v'_{3m}, v'_{3m+1} \} \]

\[ E \leftarrow \{ e_1, e_2, \ldots, e_{4m}, e_{4m+1}, e'_1, e'_2, e'_3, \ldots, e'_{4m} \} \]

\[ v_1 \leftarrow 0, v'_1 \leftarrow 0 \]

for \( i = 0 \) to \((m-1)/3\) do

\[ v_{2+9i} \leftarrow 6; v_{3+9i} \leftarrow 7; v_{4+9i} \leftarrow 4; v'_{2+9i} \leftarrow 3; v'_{3+9i} \leftarrow 1; v'_{4+9i} \leftarrow 4 \]

end for

for \( i = 0 \) to \((m-2)/3\) do

\[ v_{5+9i} \leftarrow 1; v_{6+9i} \leftarrow 0; v_{7+9i} \leftarrow 8; v'_{5+9i} \leftarrow 7; v'_{6+9i} \leftarrow 5; v'_{7+9i} \leftarrow 8 \]

end for

for \( i = 0 \) to \((m-3)/3\) do

\[ v_{8+9i} \leftarrow 1; v_{9+9i} \leftarrow 3; v'_{8+9i} \leftarrow 0; v'_{9+9i} \leftarrow 5; v'_{10+9i} \leftarrow 11 \]

end for

end procedure

Theorem 3.2: The extended duplicate graph of Quadrilateral snake graph admits \([L(3,1)]\)-labeling and its number \(\lambda(G)\) is 11.

Proof: Let QSm be the quadrilateral snake graph and EDG(QSm) be the extended duplicate graph of quadrilateral snake graph.

Define the set of vertices and edges are

\[ V(G) = \{ v_1, v_2, v_3, \ldots, v_{3m}, v_{3m+1}, v'_1, v'_2, v'_3, \ldots, v'_{3m}, v'_{3m+1} \} \]

\[ E(G) = \{ e_1, e_2, e_3, \ldots, e_{4m}, e_{4m+1}, e'_1, e'_2, e'_3, \ldots, e'_{4m} \} \]

Assume that \( V(G) = V_1 \cup V_2 \).

where \( V_1 = \{ v_i \mid 1 \leq i \leq 3m+1 \} \)

\[ V_2 = \{ v'_i \mid 1 \leq i \leq 3m+1 \} \]

For \( V_1 \) and \( V_2 \), we define a mapping \( f: V(G) \rightarrow N \cup \{0\} \) such that \(|f(x) - f(y)| \geq 3\) if \( d(x,y) = 1 \) and

\[ |f(x) - f(y)| \geq 1 \] if \( d(x,y) = 2 \).

Using the algorithm 2, the vertices \( v_1 \) and \( v'_1 \) receive the label 0;

i) \( f(v_{2+9i}) = 0 \) and \( f(v'_{2+9i}) = 3 \) for \( 1 \leq i \leq (m-1)/3 \)

ii) \( f(v_{3+9i}) = 7 \) and \( f(v'_{3+9i}) = 1 \) for \( 1 \leq i \leq (m-1)/3 \)

iii) \( f(v_{4+9i}) = 4 \) and \( f(v'_{4+9i}) = 4 \) for \( 1 \leq i \leq (m-1)/3 \)

iv) \( f(v_{5+9i}) = 1 \) and \( f(v'_{5+9i}) = 7 \) for \( 1 \leq i \leq (m-2)/3 \)
v) \( f(v_{6+9i}) = 0 \) and \( f(v'_{6+9i}) = 5 \) for \( 1 \leq i \leq (m-2)/3 \)

vi) \( f(v_{7+9i}) = 8 \) and \( f(v'_{7+9i}) = 8 \) for \( 1 \leq i \leq (m-2)/3 \)

vii) \( f(v_{8+9i}) = 1 \) and \( f(v'_{8+9i}) = 0 \) for \( 1 \leq i \leq (m-3)/3 \)

viii) \( f(v_{9+9i}) = 3 \) and \( f(v'_{9+9i}) = 5 \) for \( 1 \leq i \leq (m-3)/3 \)

ix) \( f(v_{10+9i}) = 11 \) and \( f(v'_{10+9i}) = 11 \) for \( 1 \leq i \leq (m-3)/3 \)

Thus all the vertices are labeled.

Now to prove that \( L(3,1) \) labeling number \( \lambda(G) \) is 11.

Case 1: Let \( x,y \) be any two vertices in \( V_1(G) \).

**Subcase (i):** For \( m = 3n-2, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \), such that \( x = v_{2+9i} \), \( y = v_{3+9i} \) then \( f(x) = 6 \), \( f(y) = 7 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |6 - 7| = 2 + 1 = 3 \geq 3 \).

**Subcase (ii):** For \( m = 3n-1, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \), such that \( x = v_{5+9i} \), \( y = v_{6+9i} \) then \( f(x) = 1 \), \( f(y) = 0 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |1 - 0| = 2 + 1 = 3 \geq 3 \).

**Subcase (iii):** For \( m = 3n, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_1(G) \), such that \( x = v_{8+9i} \), \( y = v_{9+9i} \) then \( f(x) = 1 \), \( f(y) = 3 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |1 - 3| = 2 + 2 = 4 \geq 3 \).

Case 2: Let \( x,y \) be any two vertices in \( V_2(G) \).

**Subcase (i):** for \( m = 3n-2, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \), such that \( x = v'_{2+9i} \), \( y = v'_{3+9i} \) then \( f(x) = 3 \), \( f(y) = 1 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |3 - 1| = 2 + 2 = 4 \geq 3 \).

**Subcase (ii):** for \( m = 3n-1, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \), such that \( x = v'_{5+9i} \), \( y = v'_{6+9i} \) then \( f(x) = 7 \), \( f(y) = 5 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |7 - 5| = 2 + 2 = 4 \geq 3 \).

**Subcase (iii):** for \( m = 3n, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two non-adjacent vertices on \( V_2(G) \), such that \( x = v'_{8+9i} \), \( y = v'_{9+9i} \) then \( f(x) = 0 \), \( f(y) = 5 \) and \( d(x,y) = 2 \). Therefore \( d(x,y) + |f(x) - f(y)| = 2 + |0 - 5| = 2 + 5 = 7 \geq 3 \).

Case 3: Let \( x \) and \( y \) be any two vertices in \( V_1(G) \) and \( v_2(G) \).

**Subcase (i):** for \( m = 3n-2, n \in \mathbb{N} \).

Let \( x \) and \( y \) be any two adjacent vertices on \( V_1(G) \) and \( V_2(G) \), such that \( x = v_{2+9i} \), \( y = v'_{3+9i} \) then \( f(x) = 6 \), \( f(y) = 3 \) and \( d(x,y) = 1 \). Therefore \( d(x,y) + |f(x) - f(y)| = 1 + |6 - 3| = 1 + 3 = 4 \geq 3 \).
Subcase (ii): for \( m = 3n-1, n \in \mathbb{N}. \)

Let \( x \) and \( y \) be any two adjacent vertices on \( V_1(G) \) and \( V_2(G) \), such that \( x = v_{5+9i} \), \( y = v'_{6+9i} \) then \( f(x)=1, f(y)=5 \) and \( d(x,y) = 1. \) Therefore \( d(x,y) + |f(x)-f(y)| = 1+|1-5| = 1+4 = 5 \geq 3. \)

Subcase (iii): for \( m = 3n, n \in \mathbb{N}. \)

Let \( x \) and \( y \) be any two adjacent vertices on \( V_1(G) \) and \( V_2(G) \), such that \( x = v_{8+9i} \), \( y = v'_{9+9i} \) then \( f(x)=1, f(y)=5 \) and \( d(x,y) = 1. \) Therefore \( d(x,y) + |f(x)-f(y)| = 1+|1-5| = 1+4 = 5 \geq 3. \)

Thus, by continuing this process of \( x \) and \( y \), we get

\[
d(x,y) + |f(x)-f(y)| = \begin{cases} 
\geq 3 & \text{if } d = 2 \\
\geq 4 & \text{if } d = 1
\end{cases}
\]

Hence, the extended duplicate graph of quadrilateral snake graph admits \( L(3,1) \)-labeling and its number \( \lambda(G) \) is 11.

Example 2: \( L(3,1) \)-labeling diagram in \( \text{EDG}(QS_4) \) and \( \text{EDG}(QS_5) \) is shown in figures (3) & (4)
4. Conclusion

In this paper, we have presented algorithms and proved that the extended duplicate graph of triangular snake graph and quadrilateral snake graph admits $L(3,1)$-Labeling.

References