Cospectral Graph Representation using Laplacian Adjacency

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Abstract

This paper attempts to study **Cospectral graph** and the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated with the graph, such as its adjacency matrix or Laplacian matrix. Two graphs are called cospectral or isospectral if the adjacency matrices of the graphs are isospectral, that is, if the adjacency matrices have equal multisets of eigenvalues. The Laplacian matrix, sometimes also called the admittance matrix (Cvetković *et al*. 1998, Babić *et al*. 2002) or Kirchhoff matrix, of a graph G, where $G = (V, E)$ is an undirected, unweighted graph without graph loops (i, i) or multiple edges from one node to another, V is the vertex set, $n = |V|$, and E is the edge set, is an $n \times n$ symmetric matrix with one row and column for each node defined by

$$\mathsf{L} = \mathsf{D} - \mathsf{A},$$

where $\mathsf{D} = \mathrm{diag}(d_1, \ldots, d_n)$ is the degree matrix, which is the diagonal matrix formed from the vertex degrees and A is the adjacency matrix. The diagonal elements $l_{i\,j}$ of L are therefore equal the degree of vertex v_i and off-diagonal elements $l_{i\,j}$ are -1 if vertex v_i is adjacent to v_j and 0 otherwise.

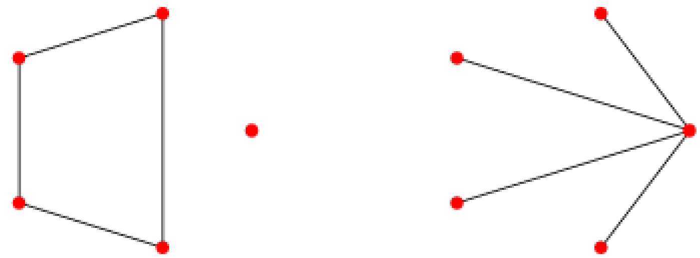
A normalized version of the Laplacian matrix, denoted $\mathcal{L}$, is similarly defined by

$$\mathcal{L}_{i\,j}(G) = \begin{cases} 1 & \text{if } i = j \text{ and } d_j \neq 0 \\ -\dfrac{1}{\sqrt{d_i\, d_j}} & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

The Laplacian matrix is a discrete analog of the Laplacian operator in multivariable calculus and serves a similar purpose by measuring to what extent a graph differs at one vertex from its values at nearby vertices. The Laplacian matrix arises in the analysis of random walks and electrical networks on graphs (Doyle and Snell 1984), and in particular in the computation of resistance distances. The Laplacian also appears in the matrix tree theorem.

Key words: cospectral; graph invariant; spectrum; Laplacian adjacency; adjacency matrix

Introduction



Cospectral graphs, also called isospectral graphs, are graphs that share the same graph spectrum. The smallest pair of isospectral graphs is the graph union $C_4 \cup K_1$ and star graph S_5, illustrated above, both of which have graph spectrum $(-2) \times 0^3 \times 2$ (Skiena 1990, p. 85). The first example was found by Collatz and Sinogowitz (1957) (Biggs 1993, p. 12). Many examples are given in Cvetkovic *et al.* (1998, pp. 156-161) and Rücker *et al.* (2002). The smallest pair of cospectral graphs is the graph union $C_4 \cup K_1$ and star graph S_5, illustrated above, both of which have graph spectrum $(-2) \times 0^3 \times 2$ (Skiena 1990, p. 85).

The following table summarizes some prominent named cospectral graphs.

n	cospectral graphs
12	6-antiprism graph, quartic vertex-transitive graph Qt19
16	Hoffman graph, tesseract graph
16	(4,4)-rook graph, Shrikhande graph
25	25-Paulus graphs
26	26-Paulus graphs
28	Chang graphs, 8-triangular graph
70	Harries graph, Harries-Wong graph

Determining which graphs are uniquely determined by their spectra is in general a very hard problem. Only a small fraction of graphs are known to be so determined, but it is conceivable that *almost all* graphs have this property (van Dam and Haemers 2003).

The total number of n-node simple graphs that are isospectral to at least one other graph on n nodes for $n = 1$, 2, ... are 0, 0, 0, 0, 1, 6, 110, 1722, 51039, ... (OEIS A099883). The numbers of pairs of isospectral simple graphs (excluding pairs that are parts of triples, etc.) are 0, 0, 0, 0, 1, 5, 52, 771, 21025, ... (OEIS A099881). Similarly, the numbers of triples of isospectral graphs (excluding triples that are parts of quadruples, etc.) are 0, 0, 0, 0, 0, 0, 2, 52

A connected graph on $n > 1$ nodes satisfies

$$\sum_{i=1}^{n} \rho(v_i) \geq \frac{1}{2}(n-1),$$

where $\rho(v_i)$ is the vertex degree of vertex i (and where the inequality can be made strict except in the case of the singleton graph K_1). However while this condition is necessary for a graph to be connected, it is not sufficient; an arbitrary graph satisfying the above inequality may be connected or disconnected.

The number of n-node connected unlabeled graphs for $n = 1$, 2, ... are 1, 1, 2, 6, 21, 112, 853, 11117, 261080, ... (OEIS A001349). The *total* number of (not necessarily connected) unlabeled n-node graphs is given by the Euler transform of the preceding sequence, 1, 2, 4, 11, 34, 156, 1044, 12346, ... (OEIS A000088; Sloane and Plouffe 1995, p. 20). Furthermore, in general, if a_n is the number of unlabeled connected graphs on n nodes satisfying some property, then the Euler transform b_n is the total number of unlabeled graphs (connected or not) with the same property. This application of the Euler transform is called Riddell's formula.

The numbers of connected labeled graphs on n-nodes are 1, 1, 4, 38, 728, 26704, ... (OEIS A001187), and the *total* number of (not necessarily connected) labeled n-node graphs is given by the exponential transform of the preceding sequence: 1, 2, 8, 64, 1024, 32768, ... (OEIS A006125; Sloane and Plouffe 1995, p. 19).

Objective:

This paper intends to explore and analyze **set of graphs called cospectral graphs,** with their adjacency matrices that must have the same characteristic polynomial also distinct characteristic polynomials, and number of **graphs** with a **cospectral** mate for the adjacency matrix

Spectral graphs

Spectral graph theory deals with the relation between the structure of a graph and the eigenvalues (spectrum) of an associated matrix, such as the adjacency matrix A and the Laplacian matrix L. Important types of relations are the spectral characterization. These are conditions in terms of the spectrum of A or L, which are necessary and sufficient for certain graph properties. Two famous examples are: (i) a graph is bipartite if and only if the spectrum of A is invariant under multiplication by −1, and (ii) the number of connected components of a graph is equal to the multiplicity of the eigenvalue 0 of L. Properties that are characterized by the spectrum for A as well as for L are the number of vertices, the number of edges, and regularity. If a graph is regular, the spectrum of A follows from the spectrum of L, and vice versa. This implies that for both A and L the properties of being regular and bipartite, and being regular and connected are characterized by the spectrum. The vertex-connectivity κ(Γ) of a graph Γ is the minimum number of vertices one has to delete from Γ such that the graph becomes disconnected. The edgeconnectivity κ ′ (Γ) is the minimum number of edges one has to delete from Γ to make the graph disconnected. One easily has that κ(Γ) ≤ κ ′ (Γ) ≤ δ(Γ) where δ(Γ) is the minimal degree of Γ. Clearly, κ(Γ) = 0 as well as κ ′ (Γ) = 0 just means that Γ is disconnected, therefore these two properties are characterized by the spectrum when Γ is regular. Fiedler [6] showed that the second smallest eigenvalue of the Laplacian matrix L (called the algebraic connectivity) is a lower bound for the vertex- (and edge-) connectivity. For a regular graph Γ there exist stronger spectral bounds for κ(Γ) (see [1]) and κ ′ (Γ) (see [4]). Here we show that for the vertex- and for the edge-connectivity in a connected regular graph there is in general no spectral characterization. For k ≥ 2 we present a pair of regular cospectral graphs Γ and Γ′ of degree 2k and order 6k, where κ(Γ) = 2k and κ(Γ′) = k+1. The edge-connectivity turned out to be much harder. Nevertheless, for every even k ≥ 4 we found a pair of regular cospectral graphs Γ and Γ′ of degree 3k − 5, where κ ′ (Γ) = 3k − 5 and κ ′ (Γ′) = 3k − 6.

The set of graph eigenvalues of the adjacency matrix is called the spectrum of the graph. (But note that in physics, the eigenvalues of the Laplacian matrix of a graph are sometimes known as the graph's spectrum.) The spectrum of a graph G with n_i-fold degenerate eigenvalues λ_i is commonly denoted $\text{Spec}(G) = (\lambda_1)^{n_1} \ (\lambda_2)^{n_2} \ \cdots$ (van Dam and Haemers 2003) or $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots \\ n_1 & n_2 & \cdots \end{pmatrix}$ (Biggs 1993, p. 8; Buekenhout and Parker 1998).

spectrum of a graph

The product $\prod_k (x - s_k)$ over the elements of the spectrum of a graph G is known as the characteristic polynomial of G, and is given by the characteristic polynomial of the adjacency matrix of G with respect to the variable x.

The largest absolute value of a graph's spectrum is known as its spectral radius.

The spectrum of a graph may be computed in the Wolfram Language using Eigenvalues[AdjacencyMatrix[*g*]]. Precomputed spectra for many named graphs can be obtained using GraphData[*graph*, "Spectrum"].

A graph whose spectrum consists entirely of integers is known as an integral graph.

The maximum vertex degree of a connected graph G is an eigenvalue of G iff G is a regular graph.

Two nonisomorphic graphs can share the same spectrum. Such graphs are called cospectral. There seems to be no standard name for graphs known to be uniquely determined by their spectra. While they could conceivably be called spectrally unique, the term "determined by spectrum" has been used in practice (van Dam and Haemers 2003).

Two nonisomorphic graphs can share the same graph spectrum, i.e., have the same eigenvalues of their adjacency matrices. Such graphs are called cospectral. For example, the graph union $C_4 \cup K_1$ and star graph S_5, illustrated above, both have spectrum $(-2, 0, 0, 0, 2)$ (Skiena 1990, p. 85). This is the smallest pair of simple graphs that are cospectral. Determining which graphs are uniquely determined by their spectra is in general a very hard problem.

Only a small fraction of graphs are known to be so determined, but it is conceivable that almost all graphs have this property (van Dam and Haemers 2002).

In the Wolfram Language, graphs known to be determined by their spectra are identified as GraphData["DeterminedBySpectrum"].

The numbers of simple graphs on $n = 1$, 2, ... nodes that are determined by spectrum are 1, 2, 4, 11, 32, 146, 934, 10624, 223629, ... (OEIS A178925), while the corresponding numbers not determined by spectrum are 0, 0, 0, 0, 2, 10, 110, 1722, 51039, 2560606, ... (OEIS A06608).

Graphs that are known to be uniquely determined by their spectra include complete graphs K_n, regular complete bipartite graphs $K_{n,n}$, cycle graphs, triangular graphs for $n \neq 8$, and the rook graphs L_k for $k \neq 4$ (Haemers 2006). In addition, the Coxeter graph, Biggs-Smith graph, collinearity graphs of the generalized octagons of orders $(2, 1)$, $(3, 1)$, and $(4, 1)$, the generalized dodecagon $(2, 1)$, the M22 graph, and the coset graphs of the doubly truncated binary Golay code and the extended ternary Golay code are determined by their spectra (van Dam and Haemers 2003b).

The complement of a distance-regular graph that is determined by its spectrum is also determined by its spectrum (van Dam and Haemers 2003b). The disjoint union of multiple copies of a strongly regular determined-by-spectrum graph is also determined by spectrum (van Dam and Haemers 2003b).

An infinite family of determined-by-spectrum graphs is given by $\mathrm{Ci}_{5k}(1, 4, 6, 9, 11, 14, \ldots, \lfloor 5k/2 \rfloor)$, which is $C_5 \otimes J_k$, where J_k is the $k \times k$ unit matrix, $\otimes$ denotes the Kronecker product of adjacency matrices (van Dam and Haemers 2003b), and 1, 4, 6, 9, 11, ... (OEIS A047209) is the sequence of positive integers that are congruent to 1 and 4 (mod 5).

Graphs that are not determined by their spectra include the rook graph L_4 and Shrikhande graph, tesseract graph Q_4 and Hoffman graph, triangular graph T_8 and Chang graphs, and the 25- and 26-Paulus graphs.

The Hoffman graph is the bipartite graph on 16 nodes and 32 edges illustrated above that is cospectral to the tesseract graph Q_4 (Hoffman 1963, van Dam and Haemers 2003). Q_4 and the Hoffman graph are therefore not determined by their spectrum. Its girth, graph diameter, graph spectrum, and characteristic polynomial are the same as those of Q_4, but its graph radius is 3 compared to the value 4 for Q_4.

The Hoffman graph has adjacency matrix given by

$$\mathrm{A} = \begin{vmatrix} 0 & \mathrm{D} \\ \mathrm{D}^T & 0 \end{vmatrix},$$

where

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{vmatrix}.$$

The Hoffman graph is an integral graph with graph spectrum $(-4)^1\ (-2)^4\ 0^6 \times 2^4 \times 4^1$.

Laplacian matrix

Spectral graph theory examines relationships between the structure of a graph and the eigenvalues (or spectrum) of a matrix associated with that graph. Different matrices are able to give different information, but all the common matrices have limitations. This is because there are graphs which have the same spectrum for a certain matrix but different structure–such graphs are called cospectral with respect to that matrix [4].

Cospectral graphs for the adjacency matrix (see for example [8,10,11,12,13]) and the Laplacian matrix (see for example, [12,17,19]) have been studied extensively, particularly for graphs with few vertices. But little is also known about cospectral graphs with respect to the normalized Laplacian since the normalized Laplacian is a rather new tool which has rather recently (mid 1990's) been popularized by Chung [7]. One of the original motivations for defining the normalized Laplacian was to be able to deal more naturally with non-regular graphs. In some situations the normalized Laplacian is a more natural tool that works better than the adjacency matrix or Laplacian matrix. In particular, when dealing with random walks, the normalized Laplacian is a natural choice. This is because D(G)−1A(G)D(G)−1A(G) is the transition matrix of a Markov chain which has the same eigenvalues as I−L(G)I−L(G). Previously, the only cospectral graphs with respect to normalized Laplacian were bipartite (complete bipartite graphs [19] and bipartite graphs found by "unfolding" a small bipartite graph in two ways [3]). Some recent studies on cospectral graphs were carried out in [1,2,5,6,14,15,16,18].

Conclusion

Construction of non-isomorphic cospectral graphs is a nontrivial problem in spectral graph theory especially for large graphs. In this paper, we establish that graph theoretical partial transpose of a graph is a potential tool to create non-isomorphic cospectral graphs by considering a graph as a partitioned graph. Two non-isomorphic graphs are said to be cospectral with respect to a given matrix if they have the same eigenvalues. Cospectral graphs help to show the limitations that the spectrum of a particular matrix might have in distinguishing properties of a graph. There are several different matrices that are used in spectral graph theory, and these different matrices can reveal different information about a graph. So graphs may be cospectral with respect to some matrix but not cospectral with respect to another matrix (though there are graphs which are cospectral with respect to all matrices). Some of the common matrices that are studied include the adjacency matrix A, the combinatorial Laplacian L := D − A, the signless Laplacian Q := D + A, the normalized Laplacian L := D−1/2 (D − A)D−1/2 (with the convention that if a vertex is isolated then the corresponding entry id D−1/2 is 0), and the Seidel matrix S := J − I − 2A. The Seidel matrix is not as commonly studied and is defined by putting a −1 for each edge, a 1 for each non-edge, and a 0 on the diagonal entries.

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