SPECTRAL THEORY FOR SELF -ADJOINT OPERATORS IN $\Gamma$-HILBERT SPACE

1 Sahin Injamamul Islam, 2 Ashoke Das

1Department of Mathematics, Raiganj University, Raiganj, West Bengal, India.
2 Associate Professor, Department of Mathematics, Raiganj University, Raiganj, West Bengal, India.

Abstract
Spectral theorem provides Spectral decomposition, Eigen Value decomposition of the vector spaces on which the operator acts. We discuss the Hilbert-Schmidt operator and function of an operator in $\Gamma$-Hilbert Space. We will show the generalization of operators explicitly in $\Gamma$-Hilbert Space while operator being self-adjoint and compact.

Key Words

1 INTRODUCTION AND PRELIMINARIES
Spectral theory for a self-adjoint operator is very important part of $\Gamma$-Hilbert Space. The definition of $\Gamma$-Hilbert Space was first introduced by Bhattacharya D.K. and T.E. Aman in their paper “$\Gamma$-Hilbert Space and linear quadretic control problem” in 2003 [1].

We shall denote $\Gamma$-Hilbert Space by $H_{\Gamma}$. Consider an operator $T: H_{\Gamma} \rightarrow H_{\Gamma}$. Spectral Theorem is a generalization of the familiar theorem from Linear algebra call that a self adjoint n x n matrix A can be diagonalized, there is a diagonal matrix D and unitary matrix U such that

$$A = UDU^{-1}.$$ 

Before Studying the Spectral theory for compact self-adjoint operator we have to reconstruct the Hilbert-Schmidt theorem. Also we present the Spectral theorem for Functional Calculus form. Here we consider both finite dimensional spaces and infinite dimensional spaces. At first we re call some important definitions:

Definition 1.1: Let E, $\Gamma$ be two linear space over the field $F$. A mapping $(\cdot, \cdot, \cdot): E \times \Gamma \times E \rightarrow \mathbb{R}$ is called a $\Gamma$-Inner product on $E$ if

(i) $(\cdot, \cdot, \cdot)$ is linear in each variable.
(ii) $\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle \ \forall \ u, v \in E \ and \ \gamma \in \Gamma$.
(iii) $\langle u, \gamma, u \rangle > 0 \ \forall \ \gamma \neq 0 \ and \ u \neq 0$.
(iv) $\langle u, \gamma, v \rangle = 0$ if atleast one of $u, v, \gamma$ is zero.

$[(E, \Gamma), (\cdot, \cdot, \cdot)]$ is called a $\Gamma$-inner product space over $F$.

A complete $\Gamma$-inner product space is called $\Gamma$-Hilbert Space.

Using the $\Gamma$-Inner product, we may define three types of norm in a $\Gamma$-Hilbert Space, namely

(i) $\gamma$-Norm
(ii) $\Gamma_{inf}$-Norm
\( y \)-norm 1.2 : Now if we write \( \|u\|_y^2 = \langle u, y, u \rangle \), for \( u \in H \) and \( y \in \Gamma \) then \( \|u\|_y^2 \) satisfy all the Conditions of norm.

\[ \|u\|_{\Gamma_{inf}} \text{-Norm} : \] If we define \( \|u\|_{\Gamma_{inf}} = \inf\{\|u\|_y : y \in \Gamma\} \). Clearly \( \Gamma_{inf} \)-Norm satisfy all the conditions of the norm for \( u \in H \).

\( \Gamma \)-Norm 1.4 : If we if write \( \|u\|_\Gamma =\{\|u\|_y : y \in \Gamma\} \) then this Norm is called the \( \Gamma \)-Norm of the \( \Gamma \)-Hilbert space.

\( y \)-Orthogonal 1.5 : Let L be a non-empty subset of a \( \Gamma \)-Hilbert space \( H_\Gamma \). Two elements \( x \) and \( y \) are said to be \( y \)-Orthogonal if their inner product \( \langle x, y, y \rangle = 0 \). In symbol, we write \( x \perp y \).

Self-adjoint operator 1.6 : Let \( A \) be a bounded operator on \( \Gamma \)-Hilbert space and we denote it by \( H_\Gamma \). Then the operator \( A^* : H_\Gamma \to H_\Gamma \) defined by

\[ \langle Ax, y, y \rangle = \langle x, y, A^*y \rangle \quad \forall x, y \in H_\Gamma \text{ and } y \in \Gamma \]

is called the adjoint operator of \( A \).

If \( A = A^* \) then \( A \) is called self adjoint of \( H_\Gamma \).

Compact Operator 1.7 : An operator \( A \) on a \( \Gamma \)-Hilbert space \( H_\Gamma \) is called a compact operator or continuous operator if, for every bounded sequence \( (x_n) \) in \( H_\Gamma \), the sequence \( (Ax_n) \) contains a convergent subsequence.

Orthonormal sequence 1.8 : A sequence \( (x_n) \) in \( H_\Gamma \) is said to be orthonormal sequence whose terms form an orthonormal set i.e it follows two conditions:

- a) \( \langle x, y, y \rangle = 0 \quad \forall x, y \in H_\Gamma \text{ and } y \in \Gamma \).
- b) \( \langle x, y, x \rangle = 1 \quad \forall x, y \in H_\Gamma \text{ and } y \in \Gamma \).

Eigenvalue 1.9 : Let \( A \) be an operator on a complex vector space \( E \). A complex number \( \lambda \) is called an eigenvalue of \( A \) if there is a nonzero vector \( u \in E \) such that

\[ Au = \lambda u \]

Eigen Spectrum 1.10: The set of all Eigen values of \( A \) is called Eigen spectrum of \( A \). It is denoted by \( \sigma_{eig}(A) \) that is \( \sigma_{eig}(A) \) of \( A \) consists of all \( k \) in \( K \) such that \( A - kI \) is not injective. Thus \( k \in \sigma_{eig}(A) \) if and only if there is some nonzero \( x \) in \( X \) such that \( A(x) = kx \).

2 RESULTS AND DISCUSSION

At first we reconstruct the Hilbert-Schmidt theorem for \( \Gamma \)-Hilbert space \( H_\Gamma \). Let \( H_\Gamma \) be a finite dimensional space. It is known from linear algebra that eigenvectors of a self-adjoint operator \( H_\Gamma \) form an orthogonal basis of \( H_\Gamma \). The following theorems generalized this result to infinite dimensional spaces.

Theorem 2.1 : For every compact, self-adjoint operator \( A \) on an infinite dimensional \( \Gamma \)-Hilbert space \( H_\Gamma \), there exists an orthonormal system of eigenvectors \( (u_n) \) corresponding to nonzero eigen value \( (\lambda_n) \) such that every element \( x \in H_\Gamma \) has a unique representation in the form

\[ x = \sum_{n=1}^{\infty} \alpha_n u_n + v, \]

Where \( \alpha_n \in \mathbb{C} \) and \( v \in N(A) \).

Proof : We have known that there exist an eigen value \( \lambda_1 \) of \( A \) such that –
\[ |\lambda_1| = \sup_{|x| \leq 1} |\langle Ax, y, x \rangle| \] \quad \text{where } x \in H_\Gamma \text{ and } y \in \Gamma.

Let \( u_1 \) be a normalized eigenvector corresponding to \( \lambda_1 \). We take
\[
Q_1 = \{ x \in H_\Gamma : x \perp y \ u_1 \},
\]
\( Q_1 \) is the orthogonal complement of the set \( \{ u_1 \} \). Thus, \( Q_1 \) is a closed linear subspace of \( H_\Gamma \). If \( x \in Q_1 \), then we have
\[
\langle Ax, y, u_1 \rangle = \langle x, y, Au_1 \rangle = \lambda_1 \langle x, y, u_1 \rangle = 0.
\]
Which means that \( x \in Q_1 \) implies \( Ax \in Q_1 \). So, \( A \) maps the \( \Gamma \)-Hilbert space \( Q_1 \) into itself. Again we apply the previous proceeding, it gives an eigenvalue \( \lambda_2 \) such that
\[
|\lambda_2| = \sup_{|x| \leq 1} |\langle Ax, y, x \rangle| : x \in Q_1 \}.
\]
Let \( u_2 \) be the normalized eigenvector of \( \lambda_2 \). Then clearly \( u_1 \perp y \ u_2 \). Next we set
\[
Q_2 = \{ x \in Q_1 : x \perp y \ u_2 \},
\]
and repeat the above argument. Having eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and the corresponding normalized eigenvectors \( u_1, u_2, \ldots, u_n \), we define
\[
Q_n = \{ x \in Q_{n-1} : x \perp y \ u_n \}
\]
Also choose an eigenvalue \( \lambda_{n+1} \) such that
\[
|\lambda_{n+1}| = \sup_{|x| \leq 1} |\langle Ax, y, x \rangle| : x \in Q_n \}.
\]
For \( u_{n+1} \) we choose a normalized vector corresponding to \( \lambda_{n+1} \).

Indeed, it can happen that there is a positive integer \( k \) such that \( \langle Ax, y, x \rangle = 0 \) for every \( x \in Q_k \) and \( y \in \Gamma \). Then every element \( x \) of \( H_\Gamma \) has a unique representation
\[
x = \alpha_1 u_1 + \cdots + \alpha_k u_k + v,
\]
Where \( Av = 0 \) and \( Ax = \lambda_1 \alpha_1 u_1 + \cdots + \lambda_k \alpha_k u_k \)
Which proves the theorem in this case.

Now suppose that the described procedure yields an infinite sequence of eigenvalues \( (\lambda_n) \) and eigenvectors \( (u_n) \). Let \( S \) be the closed space spanned by the vectors that is,
\[
S = \{ \sum_{n=1}^{\infty} \alpha_n u_n : \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \}.
\]
For every \( x \in H_\Gamma \) has a unique decomposition \( x = u + v \) or
\[ x = \sum_{n=1}^{\infty} \alpha_n u_n + v, \]

Where \( v \in s^+ \). It remains to prove that \( Av = 0 \) for all \( v \in s^+ \).

Let \( v \in s^+ \), \( v \neq 0 \). Define \( w = \frac{v}{\|v\|} \). Then

\[ \langle Av, y, v \rangle = \|v\|^2 \langle Aw, y, w \rangle. \]

Since \( w \in s^+ \subseteq Q_n \) for \( n \in \mathbb{N} \), by (i) we have-

\[ \langle Av, y, v \rangle = \|v\|^2 \langle Aw, y, w \rangle = \|v\|^2 \sup_{\|x\|=1} |\langle Ax, y, x \rangle| : x \in Q_n \]

\[ = \|v\|^2 |\lambda_{n+1}| \to 0. \]

This implies \( \langle Av, y, v \rangle = 0 \) for every \( v \in s^+ \) and \( y \in \Gamma \).

Therefore, \( Av = 0 \) for all \( v \in s^+ \).

**Hilbert-Schmidt Operator 2.2**: A bounded operator \( A \) on \( H_\Gamma \) is called Hilbert-Schmidt Operator if

\[ \sum_{j} \|A(v_j)\|^2 < \infty \]

for some orthonormal basis \( \{v_j\} \) for \( H_\Gamma \) and \( y \in \Gamma \).

**Theorem 2.3**: Let \( A \in BL(H_\Gamma) \) be a Hilbert-Schmidt operator. Then

(a) \( A \) is compact.

(b) \( A^* \) is a Hilbert-Schmidt operator on \( H_\Gamma \).

**Proof**: Let \( \{v_j\} \) be an orthonormal basis for \( H_\Gamma \) such that \( \sum_{j} \|A(v_j)\|^2 < \infty \).

(a) For \( x \in H_\Gamma \) and \( y \in \Gamma \), we consider its Fourier expansion

\[ x = \sum_{j} \langle x, y, v_j \rangle v_j. \]

Since \( A \) is continuous and linear, we have

\[ A(x) = \sum_{j} \langle x, y, v_j \rangle A(v_j). \]

For \( n = 1, 2, \ldots \), define

\[ A_n(x) = \sum_{j=1}^{n} \sum_{j} \langle x, y, v_j \rangle A(v_j), \quad x \in H_\Gamma \) and \( y \in \Gamma. \]

As \( A_n(x) \in \text{Span}\{A(v_1), \ldots, A(v_n)\} \) for all \( x \in H_\Gamma \) and range of \( A_n \) is finite dimensional. It follows that \( A_n \) is compact. Now for all \( x \in H_\Gamma \) and \( y \in \Gamma \), we have-

\[ \|A(x) - A_n(x)\|^2 = \|\sum_{j>n} \langle x, y, v_j \rangle A(v_j)\|^2 \]

\[ \leq (\sum_{j>n} \|\langle x, y, v_j \rangle\|\|A(v_j)\|)^2 \]

\[ \leq (\sum_{j>n} \|\langle x, y, v_j \rangle\|^2) (\sum_{j>n} \|A(v_j)\|^2) \]

\[ \leq \|x\|^2 \sum_{j>n} \|A(v_j)\|^2 \text{ [By Bessel’s inequality]} \]

Since \( \sum_{j>n} \|A(v_j)\|^2 \to 0 \) as \( n \to \infty \), we see that \( \|A - A_n\|_Y \to 0 \).

Hence \( A \) is compact.

(b) By Parseval’s formula, we have
Theorem 2.4: (Spectral theorem for compact self-adjoint operators) Let $A$ be a compact, self-adjoint operator on an infinite-dimensional $\Gamma$-Hilbert space $H_{\Gamma}$. Then $H_{\Gamma}$ has a complete orthonormal system (an orthonormal basis) $\{v_1, v_2, ..., v_n\}$ consisting of eigenvectors of $A$. Moreover, for every $x \in H_{\Gamma}$,

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n,$$

where $\lambda_n$ is the eigenvalue corresponding to $v_n$ and $\gamma \in \Gamma$.

**Proof:** To obtain a complete orthonormal system $\{v_1, v_2, ..., v_n\}$, we need to complement the system $\{u_1, u_2, ..., u_n\}$, defined in the proof of Theorem 2.1, with an arbitrary orthonormal basis of $N(A)$. The eigenvalues corresponding to the vectors that form $N(A)$ are all equal to zero. Equality $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$ follows from the continuity of $A$.

Corollary 2.5: Let $A$ be a compact self-adjoint operator on a $\Gamma$-Hilbert Space $H_{\Gamma}$. Then

$$A = \sum_{n=1}^{\infty} \lambda_n P_n$$

where $P_n$ is a projection on finite-dimensional $H_{\Gamma}$.

**Proof:** Let $\{v_1, v_2, ..., v_n\}$ be a complete orthonormal system of eigenvectors of $A$ corresponding to eigenvalues $\{\lambda_1, \lambda_2, ..., \}$. Let $P_n$ be the projection operator onto the one-dimensional space spanned by $v_n$.

Since $P_n x = \langle x, v_n \rangle v_n$, the equation $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$ can be written as $Ax = \sum_{n=1}^{\infty} \lambda_n P_n x$.

Here is another way of representing $A$ in the form $A = \sum_{n=1}^{\infty} \lambda_n P_n$. Let $\{\lambda_1, \lambda_2, ..., \}$ be all distinct nonzero eigenvalues of $A$ and let $P_n$ be the projection onto the eigenspace corresponding to $\lambda_n$. The eigenspaces corresponding to nonzero eigenvalues of a compact self-adjoint operator are finite-dimensional, those eigenspaces are finite-dimensional.

Above corollary is just another version of the Spectral theorem. This version is important because it has natural extensions to more general classes of operators. It is also useful as it leads to an elegant expression for powers and other functions of an operator.

Let $A, \lambda_n, \text{ and } P_n$ be as mentioned in the above corollary. Then

$$A^2 = A(\sum_{n=1}^{\infty} \lambda_n P_n)$$

$$= \sum_{n=1}^{\infty} \lambda_n A P_n$$

$$= \sum_{n=1}^{\infty} \lambda_n^2 P_n,$$

because $A P_n x = \lambda_n P_n x$ for every $x \in H_{\Gamma}$. Similarly, for any $k \in \mathbb{N}$, we get

$$A^k = \sum_{n=1}^{\infty} \lambda_n^k P_n.$$
and hence for any polynomial \( p(t) = \alpha_n t^n + \ldots + \alpha_1 t \), we have
\[
P(A) = \sum_{n=1}^{\infty} p(\lambda_n) P_n.
\]

The constant term in \( p \) must be zero, because otherwise the sequence \((p(\lambda_n))\) would not converge to zero. To deal polynomials with a nonzero constant term \( \alpha_0 \), we add \( \alpha_0 I \). In such case, \( P(A) \) is not a compact operator.

**Function of a operator 2.6:** Let \( f \) be a real-valued function \( \mathbb{R} \) such that \( f(\lambda) \to 0 \) as \( \lambda \to 0 \) and \( f(0) = 0 \).

For a compact, self-adjoint operator \( A \) of \( \mathcal{H}_\Gamma \), \( A = \sum_{n=1}^{\infty} \lambda_n P_n \), We define
\[
f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n.
\]

**Example 2.6.1:** Let \( A = \sum_{n=1}^{\infty} \lambda_n P_n \) be a compact, self-adjoint operator of a \( \Gamma \)-Hilbert Space \( \mathcal{H}_\Gamma \).

We can define sine of \( A \) by
\[
\sin A = \sum_{n=1}^{\infty} (\sin \lambda_n) P_n.
\]

If \( A = \sum_{n=1}^{\infty} \lambda_n P_n \) and \( P_n x = \langle x, \gamma, v_n \rangle v_n \), then for any \( x \in \mathcal{H}_\Gamma \) and \( \gamma \in \Gamma \), we have-
\[
(f(A))x = \sum_{n=1}^{\infty} f(\lambda_n) \langle x, \gamma, v_n \rangle v_n.
\]

Where the convergence of the series is justified, because
\[
|f(\lambda_n) \langle x, \gamma, v_n \rangle|^2 \leq M |\langle x, \gamma, v_n \rangle|^2
\]
for some constant \( M \), and hence \( f(\lambda_n) \langle x, \gamma, \lambda_n \rangle \in l^2 \).

Clearly, in this case, we cannot expect \( f(A) \) to be a compact operator.

**Theorem 2.6.2:** If eigenvectors \( u_1, u_2, \ldots \) of a self-adjoint operator \( T \) on a \( \Gamma \)-Hilbert Space \( \mathcal{H}_\Gamma \) form a complete \( \gamma \)-orthonormal system in \( \mathcal{H}_\Gamma \) and all eigenvalues are positive (or non-negative), then \( T \) is strictly positive (or positive).

**Proof:** Suppose \( u_1, u_2, \ldots \) is a complete \( \gamma \)-orthonormal system of eigenvalues of \( T \) corresponding to eigenvalues \( \lambda_1, \lambda_2, \ldots \). Then any non-zero vector \( u \in \mathcal{H}_\Gamma \) can be represented as \( = \sum_{n=1}^{\infty} \alpha_n u_n \), and we have
\[
\langle T u, \gamma, u \rangle = \langle T u, \gamma, \sum_{n=1}^{\infty} \alpha_n u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \alpha_n \langle T u, \gamma, u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \overline{\alpha_n} \langle u, \gamma, T u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \overline{\alpha_n} \langle u, \gamma, \lambda_n u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \alpha_n \overline{\alpha_n} \langle u, \gamma, u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \lambda_n \alpha_n \langle u, \gamma, u_n \rangle
\]
\[
= \sum_{n=1}^{\infty} \lambda_n |\alpha_n|^2 \geq 0
\]
Where $\gamma \in \Gamma$ and all eigenvalues are non-negative. If all $\lambda_n$’s are positive, then the last inequality becomes strict.

**Theorem 2.6.3**: For any two commuting, compact, self-adjoint operators $A$ and $B$ on a $\Gamma$-Hilbert Space $H_{\Gamma}$, there exists a complete orthonormal system in $H_{\Gamma}$ of common eigenvectors of $A$ and $B$.

**3 CONCLUSION**

We basically discussed the spectral representation of compact self-adjoint operators. We tried to show that Compact Self-adjoint operators Satisfies the Spectral theorem and we show the extensions of the theorem by generalized the function of operator in $\Gamma$-Hilbert Space.

**REFERENCES**


