Śulbasūtras and Approximate Value of $\sqrt{2}$

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Abstract: Śulbasūtras was the oldest mathematical text found in India. The Śulbasūtras contain rich principles of mathematics, basically of ‘geometry’. The outstanding feature of Śulbasūtras is consistency and completeness of geometrical results and application of these results in actual construction shows that Śulbasūtras have deeper significance.

Following statement in Śulbasūtras has great importance because it covers the idea of properties of square. Further it covers the idea of irrational number $\sqrt{2}$.

$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\times4} - \frac{1}{3\times4\times34}$

Meaning: - “The diagonal cord of a square makes double the area”.

Śulbasūtras stated the value of $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\times4} - \frac{1}{3\times4\times34}$. Hence this value is practically useful and reliable for the geometrical constrictions.

Keywords: Śulbasūtras, Śulbasūtraskāras, Karaṇī, Approximate value of $\sqrt{2}$

I. INTRODUCTION

Baudhāyana, Āpastamba and Kātyāyana Śulbasūtras stated the value of $\sqrt{2}$ in combining two squares of equal areas, in this case square has side $\sqrt{2}a$. The diagonal of a square of side a is $\sqrt{2}a$.

II. Approximate value of $\sqrt{2}$ in Śulbasūtras

In Indian mathematics the value of $\sqrt{2}$ was first observed in Baudhāyana and Āpastamba and Śulbasūtras. For the given square of side a, Baudhāyana and Āpastamba obtained the side d, of a square whose area is double the area of given square. This value of $\sqrt{2}$ was stated as “Increase the measurement (of which dvi-
karaṇī (side of a square whose area is double than given square) is to be found) by its third part, and again by the fourth part (of this third part) less by the thirty-fourth part of itself (i.e. of this fourth part). The value thus obtained is called saviśeṣa(approximate)".

Kātyāyana states that, If d be the dvi- karaṇī of a, that is , if d be the side of a square whose area is double that of the square whose side is a, then according to the rule, d is given by

\[ d = a + \frac{a}{3} + \frac{a}{3 \times 4} - \frac{a}{3 \times 4 \times 34} \]

More clearly the diagonal of a square is its dvi- karaṇī which is \( \sqrt{2} \) a). This rule gives us the relation between side and the diagonal of a square. Thus we get

\[ \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} \]

Expressing in decimal fractions we obtain \( \sqrt{2} = 1.4142156.... \) and the modern value is \( \sqrt{2} = 1.414213.... \) correct up to five decimal places.

The question arises: how did Baudhāyana, Āpastamba or Kātyāyana find this value of \( \sqrt{2} \). There was no method given in the Śulbasūtras but the formula \( \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} \) itself is self explanatory.

In Śulbasūtras many geometrical constructions are given. One of which is, to find a square whose area is equal to the sum (or difference) of areas of two unequal squares. Once square is constructed we can find the side of the square which gives approximate values of surds.

There was no proof is given in finding the approximate value of surd. We will use best possible hypothesis in finding the value of surd. The Śulbasūtras contain geometric construction only. We find the value of surds \( \sqrt{2} \) using geometric construction.

**III. METHOD 1 (OBTAINING A SQUARE FROM TWO SQUARES OF UNIT LENGTH)**

This is most simple and possible hypothesis. Take two squares whose sides are unit length. Divide the second square into three equal strips I, II and III of side \( \frac{1}{3} \) each. Sub-divide the last strip III into three small squares III\(_1\), III\(_2\) and III\(_3\). (This is possible because we are obtaining nine squares from the original square). Place I along one side of square S and put II along the upper side of square S. Put III\(_1\) in the corner as shown in the figure.

Now divide each of the portions III\(_2\) and III\(_3\) into four equal strips. Placing four and four of them about the square just formed along its east and north side. Now introduce a small square at the north east corner. The larger square will be formed whose side is equal to \( \left(1 + \frac{1}{3} + \frac{1}{3 \times 4}\right) \). Now the square which we obtain is larger than two original squares by an amount \( \left(\frac{1}{3 \times 4}\right)^2 \) which is the area of the small square introduce at the corner. Now suppose x is the breadth of small strip then we must have

\[ 2x \left(1 + \frac{1}{3} + \frac{1}{3 \times 4}\right) - x^2 = \left(\frac{1}{3 \times 4}\right)^2 \]

neglecting \( x^2 \) as too small we get

\[ x = \frac{1}{3 \times 4 \times 34} \]

Thus finally we have \( \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} \)
IV. Method 2 (finding the length of a diagonal of a square of side 12)

1) Finding the value of $\sqrt{2}$ by finding the side of square whose area is $2a^2$ where ‘a’ is a side of a square whose diagonal is $\sqrt{2}a$.

All the authors of Āryabhaṭa’s Śulbasūtras try to obtain a square whose side and diagonal are integers. They reached to a conclusion that such a square whose side and diagonal are integers cannot be obtain. Then what they have done.

They began by assuming two as the measure of a square’s side. Then the square of the diagonal is 8. They tried to find the side of a square whose area is 8 which is $\sqrt{8}$. But all Śulbasūtraskāras were unsuccessful.

They (Śulbasūtraskāras) tried for the next number, taking side of a square 3. The square of diagonal is 18. But they did not obtain $\sqrt{18}$.

Now we find the square of a diagonal nearer to the real square number. For that we construct a table in which first column is side of a square (suppose a), second column is square of a diagonal of square of side a, the third column is nearest square number.

<table>
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<th>a</th>
<th>$2a^2$</th>
<th>Nearest Sq</th>
<th>a</th>
<th>$2a^2$</th>
<th>Nearest Sq</th>
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<td>1</td>
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Here we have three cases in which the square of the diagonal of a square differs by 1 from a square of a number. These cases are 8-9, 50-49, 288-289. The last case is most suitable because it contain the largest number.

We can say that (for square of side 12 whose diagonal square is 288 which is nearer to 289 square of 17) the diagonal of a square of which side is 12 was very little shorter than seventeen. We try to reduce 17 in such a way that square of which is almost equal to 288.
Now suppose Śulbasūraskāras drew a square whose side was 17 padas (Unit) long. Divide this square into $17 \times 17 = 289$ small squares. Then Śulbasūraskāras shortened the side of square containing 289 small squares, so that its area contains not 289 but 288 small squares. The measure (length) of side of square containing 288 small squares, would be exact measure (length) of diagonal of square whose side is equal to 12 ($12^2 + 12^2 = 288$). Otherwise the measure (length) of the side of square containing 288 small squares would be exact square root of 288.

Now the question will arise that how Śulbasūraskāras shorten the side of a square containing 289 small squares? They cut two strips from the two sides of a square (one strip from north side and other strip from east side, as shown in figure). The width of these two strips is so that the area of these two strips is equal to one of the 289 small squares.

Suppose $x = 12$ (side of a square whose length of diagonal have to be found out). Then the length of side of one small square is $(x/12)$. When two strips are cut resulting 34 small rectangles whose are equal to one small square. Here width of rectangle is $\frac{x}{12 \times 34}$. There is error in calculating the area, of a small square in the north-east corner. Neglecting this error we calculate the length of side of square containing 288 small squares. This side is $17 - \frac{x}{12 \times 34}$. We write 17 as 12+4+1. Now the side of square containing 288 squares is $12\sqrt{2}$.

$$12\sqrt{2} = 12 + 4 + 1 - \frac{1}{34}$$ put $x = 12$ we write it as $x\sqrt{2} = x + \frac{x}{3} + \frac{x}{12} - \frac{x}{12 \times 34}$ cancelling x we have $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}$. This is the best possible value of $\sqrt{2}$ using geometry.

V. Conclusion

The value of $\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}$ which is stated in Śulbasūtras was derived from geometrical construction. Hence this value is practically useful and reliable for the geometrical constritions.

Śulbasūtras was the oldest mathematical text in India. The Śulbasūtras contain rich principles of mathematics, basically of ‘geometry’. The Śulbasūtras were considered as instruction manual for the construction of altars and fireplaces.

The important feature of Śulbasūtras is consistency and completeness of geometrical results. These results have application in actual construction shows that Śulbasūtras have deeper significance.
VI. References
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