Flow Kinematics: Local Expansion of the Velocity Field

***Basavraj.B. Assistant Professor of Mathematics, SMBK Govt. First Grade College, Naregal.**

Abstract

This paper attempts to study **Fluid kinematics in fluid mechanics** referring to a mere mathematical description or specification of a **velocity flow field**, divorced from any account of the forces and conditions that might actually create such a flow. The term fluids includes liquids or gases, but also may refer to materials that behave with fluid-like properties, including crowds of people or large numbers of grains if those are describable approximately under the continuum hypothesis as used in continuum mechanics. The composition of the material contains two types of terms: those involving the time derivative and those involving spatial derivatives. The time derivative portion is denoted as the local derivative, and represents the effects of unsteady flow. The local derivative occurs during unsteady flow, and becomes zero for steady flow.

The portion of the material derivative represented by the spatial derivatives is called the convective derivative. It accounts for the variation in fluid property, be it velocity or temperature for example, due to the motion of a fluid particle in space where its values are different. Consideration of the velocity field alone is referred to as flow field kinematics in distinction from flow field dynamics (force considerations). Fluid mechanics and especially flow kinematics is a geometric subject and if one has a good understanding of the flow geometry then one knows a great deal about the solution to a fluid mechanics problem. V can be expressed in any coordinate system; e.g., polar or spherical coordinates. Recall that such coordinates are called orthogonal curvilinear coordinates. The coordinate system is selected such that it is convenient for describing the problem at hand (boundary geometry or streamlines). The (rate of) deformation tensor is what distinguishes fluid motion from rigid body motion. The rate of deformation tensor is a symmetric tensor and the principal axis of deformation can be found. They correspond to the eigenvalues of the matrix and the eigenvalues are the principal rates of strain. A set of particles that is originally on the surface of a sphere will be deformed to an ellipsoid whose axes are coincident with the principal axis.

Key words: Vorticity , Kinematics velocity, Fluid Motion, vortex lines, and tubes.

Introduction

Kinematics of fluid flow deals with the motion of fluid particles without considering the agency producing the motion. This deals with the geometry of motion of fluid particles. This also deals with the velocity and acceleration of fluid particles in motion. The motion of a fluid can be analysed on the same principles as those applied in the motion of a solid.

There, however exists a basic difference between the motion of a solid and the motion of a fluid. A solid body is compact and moves as one mass. There is no relative motion between the particles of a solid body. Hence, we study the motion of the entire body and there is no necessity to study the motion of any particle of a solid body.

But in the case of a fluid body, the fluid particles are all separately mobile and have motions independently. A fluid particle may have a motion different from those surrounding it. However, it may be possible to obtain a relationship between the motions of neighbouring fluid particles. We know that each particle of a fluid in motion has at any instant a certain definite value of its properties like density, velocity, acceleration etc. As the fluid moves on, the values of these properties will change from one position to other positions, from time to time.

Thus, it may be realized that two methods are possible to describe fluid motion. In the first method called the Lagrangian method, we study the velocity, acceleration etc. of an individual fluid particle at every instant of time as the particle moves to different positions.

This method of studying the properties of a single fluid particle is a very tedious process and therefore this method is not generally adopted. In the second method called the Eulerian method, we describe the flow by studying the velocity, acceleration, pressure, density etc. at a fixed point in space. Due to its easy application, this method is most commonly adopted.

The continuity equation is developed based on the principle of conservation of mass. The continuity equation states that the rate of fluid flow through the pipe is constant at all cross-sections. That is, the quantity of fluid per second is constant throughout the pipe section.

Consider a fluid, flowing through a pipe with varying cross-sectional areas, as shown in figure-1 below. Consider two sections 1-1 and 2-2 as shown.

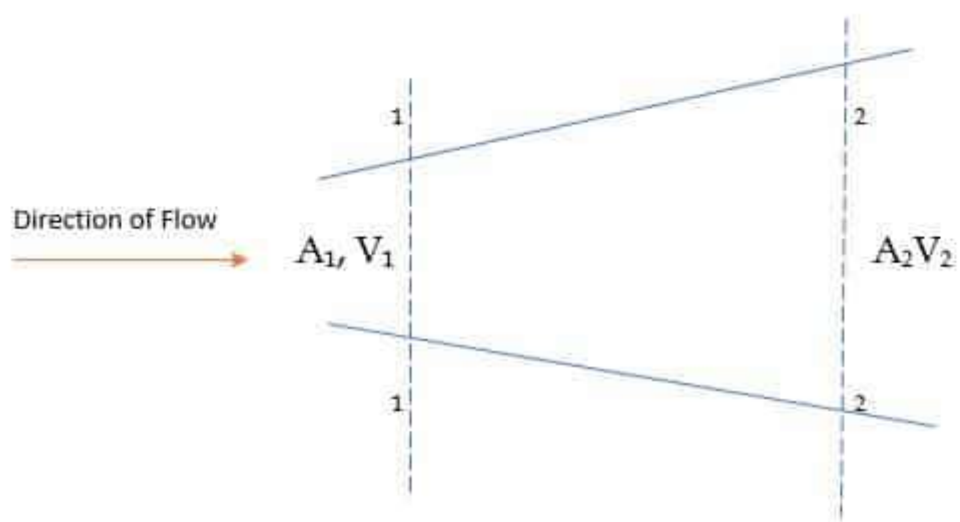


Fig.1. Continuity Equation in Fluid Mechanics

The area of the cross-section in sections 1 and 2 be A1 and A2 respectively. The velocity and density of fluid at section 1-1 be V_1 and J_1 & that of section-2-2 be V_2 and J_2. Then, from Equation-1,

Rate of Flow or Discharge at Section 1-1 , $Q_1 = J_1A_1V_1$

Rate of flow or discharge at Section 2-2 ,$Q_2 = J_2A_2V_2$

Based on the Continuity equation, the rate of flow of fluid in section 1-1 is equal to the rate of flow of fluid in section 2-2. Then,

$Q_1 = Q_2$

$J_1A_1V_1 = J_2A_2V_2$

The above equation is applicable to compressible flow (The fluid flow in which the density varies with time). For incompressible flow, the continuity equation is given by the equation,

$$A_1V_1 = A_2V_2$$

The continuity equation in cartesian coordinates can be applicable for:

1. Steady and Unsteady Fluid Flow

2. Uniform and Non-Uniform Fluid Flow

3. Compressible and Incompressible Fluid Flow

Continuity Equation for Steady Flow

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(pv) + \frac{\partial}{\partial z}(pw) = 0$$

Objective:

This paper intends to explore and analyze the **fluid flow** is described in terms of '**vector fields**' or. 'tensor fields' such as velocity, stress, pressure**.** Also **Kinematics** of geodesic **flows** on specific, two-dimensional, curved surfaces (the sphere, hyperbolic space and the torus)

Kinematics of Fluid Motion

Fluid kinematics defines the motion of the fluids and its significance without taking into account of the nature of forces that cause motion.

Three main features:

- Improvement of methods and tools for recitation and identifying the motion of fluids.
- Determination of the conditions for the kinematic option of fluid motions
- Classification of different kinds of motions and related distortion rates of any fluid component.

Scalar and vector Sector:

Scalar is a quantity which can be expressed by a particular number represents its magnitude.

In a region if every point of the scalar functions has a defined value, then that region is known as the scalar sector.

Vector is a quantity which can be expressed in both the direction and the magnitude.

In a region if every point of the vector functions has a defined value. Then that region is known as the vector sector.

velocity of the flowing fluid

Flow sector:

The area in which the flow elements such as pressure, velocity etc., which are to be defined at every point of an instant of time is known as flow sector.

Flow sectors are identified by the velocity observed at different areas in that particular region, at various times factors.

Fluid motions are divided into two types they are

- Lagrangian method
- Eulerian Method

Specification of the flow field

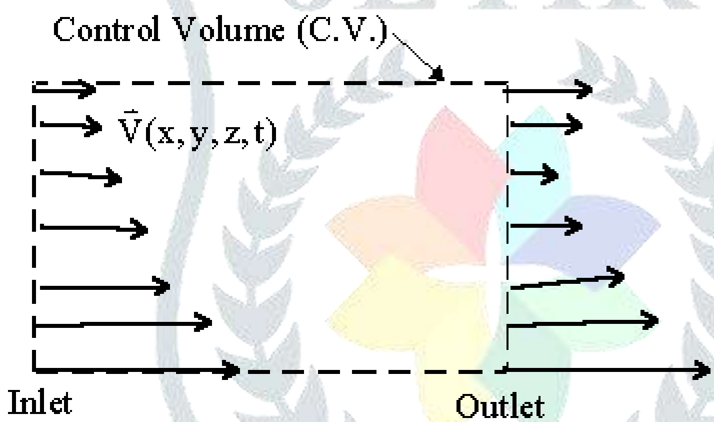
The continuum hypothesis enables us to use the simple concept of local velocity of the fluid, and we must now consider how the whole field of flow may be specified as an aggregate of such local velocities. Two distinct alternative kinds of specification are possible. The first, usually called the Eulerian type, is like the specification of an electromagnetic field in that the flow quantities are defined as functions of position in space (**x**) and time (*t*), The primary flow quantity is the (vector) velocity of the fluid, which is thus written as $\mathbf{u}(\mathbf{x}, t)$. This Eulerian specification can be thought of as providing a picture of the spatial distribution of fluid velocity (and of other flow quantities such as density and pressure) at each instant during the motion.

The second, or Lagrangian type of specification, makes use of the fact that, as in particle mechanics, some of the dynamical or physical quantities refer not only to certain positions in space but also (and more fundamentally) to identifiable pieces of matter. The flow quantities are here defined as functions of time and of the choice of a material element of fluid, and describe the dynamical history of this selected fluid element.

Lagrangian methods is used for single fluid particles

In the Lagrangian description of fluid flow, individual fluid particles are "marked," and their positions, velocities, etc. are described as a function of time. In the example shown, particles A and B have been identified. Position vectors and velocity vectors are shown at one instant of time for each of these marked particles. As the particles move in the flow field, their postions and velocities change with time, as seen in the animated diagram. The physical laws, such as Newton's laws and conservation of mass and energy, apply directly to each particle. If there were only a few particles to consider, as in a high school physics experiment with billiard balls, the Lagrangian description would be desirable. However, fluid flow is a continuum phenomenon, at least down to the molecular level. It is not possible to track each "particle" in a complex flow field. Thus, the Lagrangian description is rarely used in fluid mechanics.

- The *Eulerian Description* is one in which a control volume is defined, within which fluid flow properties of interest are expressed as *fields*.



In the Eulerian description of fluid flow, individual fluid particles are not identified. Instead, a control volume is defined, as shown in the diagram. Pressure, velocity, acceleration, and all other flow properties are described as *fields* within the control volume. In other words, each property is expressed as a function of space and time, as shown for the velocity field in the diagram. In the Eulerian description of fluid flow, one is not concerned about the location or velocity of any particular particle, but rather about the velocity, acceleration, etc. of whatever particle happens to be at a particular location of interest at a particular time. Since fluid flow is a continuum phenomenon, at least down to the molecular level, the Eulerian description is usually preferred in fluid mechanics. Note, however, that the physical laws such as Newton's laws and the laws of conservation of mass and energy apply directly to particles in a Lagrangian description. Hence, some translation or reformulation of these laws is required for use with an Eulerian description.

- Example - Pressure field - An example of a fluid flow variable expressed in Eulerian terms is the pressure. Rather than following the pressure of an individual particle, a pressure *field* is introduced, i.e. **p** = **p(x,y,z,t)**.
  Note that pressure is a *scalar*, and is written as a function of space and time (x,y,z, and t). In other words, at a

given point in space (x,y, and z), and at some particular time (t), the pressure is defined. In the Eulerian description, it is of no concern *which* fluid particle is at that location at that time. In fact, whatever fluid particle happens to be at that location at time t experiences the pressure defined above.

- Example - Velocity field - An example of a fluid flow variable expressed in Eulerian terms is the velocity. Rather than following the velocity of an individual particle, a velocity *field* is introduced, i.e.

$$\vec{V}(x,y,z,t) = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k}$$

Note that since velocity is a *vector*, it can be split into three components (u,v, and w), all three of which are functions of space and time (x,y,z, and t). In other words, at a given point in space (x,y, and z), and at some particular time (t), the velocity vector is defined. In the Eulerian description, it is of no concern *which* fluid particle is at that location at that time. In fact, whatever fluid particle happens to be at that location at time t has the velocity defined above.

- Example - Acceleration field - An example of a fluid flow variable expressed in Eulerian terms is the acceleration. Rather than following the acceleration of an individual particle, an acceleration *field* is introduced, i.e.

$$\vec{a}(x,y,z,t) = \vec{a}_x(x,y,z,t)\vec{i} + \vec{a}_y(x,y,z,t)\vec{j} + \vec{a}_z(x,y,z,t)\vec{k}.$$

Note that since acceleration is a *vector*, it can be split into three components, all three of which are functions of space and time (x,y,z, and t). In other words, at a given point in space (x,y, and z), and at some particular time (t), the acceleration vector is defined. In the Eulerian description, it is of no concern *which* fluid particle is at that location at that time. In fact, whatever fluid particle happens to be at that location at time t has the acceleration defined above.

- Either description method is valid in fluid mechanics, but the Eulerian description is usually preferred because there are simply too many particles to keep track of in a Lagrangian description.

- The *Material Derivative*, also called the *Total Derivative* or *Substantial Derivative* is useful as a bridge between Lagrangian and Eulerian descriptions.

- Definition of the material derivative - The material derivative of some quantity is simply defined as the rate of change of that quantity following a fluid particle. It is derived for some arbitrary fluid property Q as follows:

$$\begin{aligned}\frac{DQ}{Dt} = \frac{dQ}{dt} &= \frac{\partial Q}{\partial t}\frac{dt}{dt} + \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} + \frac{\partial Q}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial Q}{\partial t}(1) + \frac{\partial Q}{\partial x}(u) + \frac{\partial Q}{\partial y}(v) + \frac{\partial Q}{\partial z}(w) \\ &= \frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z}\end{aligned}$$

In this derivation, dt/dt = 1 by definition, and since a fluid particle is being followed, dx/dt = u, i.e. the x-component of the velocity of the fluid particle. Similarly, dy/dt = v, and dz/dt = w following a fluid particle.

Note that Q can be any fluid property, scalar or vector. For example, Q can be a scalar like the pressure, in which case one gets the material derivative or substantial derivative of the pressure. In other words, dp/dt is the rate of change of pressure following a fluid particle. Or, using the same equations above, Q can be the velocity vector, in which case one gets the material derivative of the velocity, which is defined as the material acceleration, i.e. the rate of change of velocity following a fluid particle.
Note also the notation, DQ/DT, which is used by some authors to emphasize that this is a material or *total* derivative, as opposed to some partial derivative. DQ/DT is identical to dQ/dt.
The material derivative is a *field* quantity, i.e. it is expressed in the Eulerian frame of reference as a function of space and time (x,y,z,t). Thus, at some given spatial location (x,y,z) and at some given time (t), DQ/Dt = dQ/dt = the material derivative of Q, and is defined as the total rate of change of Q with respect to time as one follows whatever fluid particle happens to be at that location at that instant of time. Q changes for two reasons: First, if the flow is unsteady, Q changes directly with respect to time. This is called the *local* or unsteady rate of change of Q. Second, Q changes as the fluid particle migrates or convects to a new location in the flow field. This is called the *convective* or advective rate of change of Q.

- Example - the material acceleration, following a fluid particle - The material acceleration can be derived as follows:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt}$$
$$= \frac{\partial \vec{V}}{\partial t}(1) + \frac{\partial \vec{V}}{\partial x}(u) + \frac{\partial \vec{V}}{\partial y}(v) + \frac{\partial \vec{V}}{\partial z}(w)$$
$$= \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

  Note that dt/dt = 1 by definition, and since a fluid particle is being followed, dx/dt = u, i.e. the x-component of the velocity of the fluid particle. Similarly, dy/dt = v, and dz/dt = w following a fluid particle. The first term on the right hand side is called the *local acceleration* or the *unsteady acceleration*. It is only non-zero in an unsteady flow. The last three terms make up the *convective acceleration*, which is defined as the acceleration due to convection or movement of the fluid particle to a different part of the flow field. The convective acceleration can be non-zero even in a steady flow! In other words, even when the velocity field is *not* a function of time (i.e. a steady flow), a fluid particle is still accelerated from one location to another.

- History of the method can be traced with the help of the motion and trajectory of the fluid particles.
- From the beginning of the method the particles are traced throughout the motion, where exchange of mass is essential.

Disadvantages of Lagranigian method:

- For practical applications this method is rarely used.

Eulerian Method

Eulerian method is developed by Leonhard Euler

This method is used for particular section or point.

Unsteady flow Uniform flow Non – Uniform flow:

In the unsteady flow the flow characteristics and fluid parameters are changed with the time.

At any instant of time the velocity vectors flow uniform and they do not change point to point with time, which called as uniform flow.

At any instant of time the velocity vector varies from point to point of time is known as the non – uniform flow.

$$\frac{dv}{ds} = 0$$

This flow is applicable for the velocity only.

The particles that have the random and erratic movement, intermixing with the adjacent layers is known as the turbulent flow.

When the particles move in layers sliding smoothly over the adjacent layer are known as laminar flow.

Streak lines are the locus of the points. In the past all the particles that always passes through a specific spatial point. Dye steadily inserted into the fluid at a static point spreads along a streak line.

Mathematical Description:

Streak lines are defined as:

$$\overrightarrow{u_P}$$

Is the velocity of the particle and P is the location at and the time t.

Path line:

During motion path line is a curve traced by a single fluid particle

Mathematical Description:

Path line is defined as:

P indicates the motion of a fluid particle. At the point the curve must be parallel to the flow of the velocity vector .The velocity vector is calculated at the position of the particle at that time t.

Streamline:

Streamline is defined as an imaginary line which is drawn in a flow field, such as a tangent drawn at any point. On that line it represents the direction of the velocity vector at that point.

There is no velocity component normal to the stream lines.

Streamline is defined as:

$$\frac{d\vec{x}_S}{ds}\times\vec{u}(\vec{x}_S)=0$$

where,

x= vector cross product and

$$\vec{x}_S(S)$$

= parametric representation of single streamline at single moment in time.
Velocity of the components =(u,v,w) and stream lines are expressed as

$$\overrightarrow{x_s}=(x_s,y_s,z_s)$$

it as

$$\frac{dx_s}{u}=\frac{dy_s}{v}=\frac{dz_s}{w}$$

Conclusion

Kinematics is the study of motion without regard to the forces that bring about the motion. Already, we have described how rigid body motion is described by its translation and rotation. Also, the divergence and curl of the field and values on boundaries can describe a vector field. Here we will consider the motion of a fluid as microscopic or macroscopic bodies that translate, rotate, and deform with time. We treat fluids as a continuum such that the fluid identified to be at a specific point in space at one time with neighboring fluid will be at another specific point in space at a later time with the same neighbors, with the exception of certain bifurcations. This identification of the fluid occupying a point in space requires that the motion is deterministic rather than stochastic, i.e., random motions such as diffusion and turbulence are not described. Central to the kinematics of fluid motion is the concept of convection or following the motion of a “particle” of fluid. Much of the fluid mechanics component is concerned with finding the velocity vector fields for fluids in motion. With this in mind, we first investigate ways of visualising the velocity field. Clearly, when we predict a velocity field from mathematical theory, we shall require a means of validating the results by comparison with the actual flow under investigation. The visualisation of a real fluid flow is important for the modelling process, and many new modelling ideas owe their origin to some form of flow visualisation. Two methods of visualising fluid flows relate to the **pathlines and streamlines** of the flow. The set of all positions of the particle from t = 0 to t = 3 is a continuous path. A pathline is the path traced out by an individual fluid particle during a specified time interval, , a streamline at an instant of time is a Streamlines are sometimes called flowlines or lines of flow. curve such that, at each point along the curve, the tangent vector to the curve is parallel to the fluid velocity vector.

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