ON THE EXPONENTIAL DIOPHANTINE EQUATION $439^p + 457^q = r^2$

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Abstract: Diophantine equation is the subject undergoing intense study by the researchers in number theory as these equations help in solving various advance puzzle problems. In this article, authors discussed the existence of solution of exponential Diophantine equation $439^p + 457^q = r^2$, where $p, q, r$ are whole numbers. Results show that the exponential Diophantine equation $439^p + 457^q = r^2$, where $p, q, r$ are whole numbers, has no solution in whole number.

Keywords: Prime number; Diophantine equation; Solution, Integers; Number theory.

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Introduction: Diophantine equations are those equations which are to be solved in integers. Diophantine equations are very important equations of theory of numbers and have many important applications in algebra, analytical geometry and trigonometry. Aggarwal et al. [1] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution. Aggarwal et al. [2] discussed the existence of solution of Diophantine equation $181^x + 199^y = z^2$. Bhatnagar and Aggarwal [3] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in whole number.

Gupta and Kumar [4] gave the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$. Kumar et al. [5] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [6] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [7] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution.
Mishra et al. [8] studied the existence of solution of Diophantine equation $211^α + 229^β = γ^2$ and proved that the Diophantine equation $211^α + 229^β = γ^2$ has no solution in whole number. Diophantine equations help us for finding the integer solution of famous Pythagoras theorem and Pell’s equation [9-10]. The Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [11, 14]. He proved that these equations have a unique solution which is given by $\{x = 1, y = 0, z = 3\}$. Sroysang [12] proved that the exponential Diophantine equation $31^x + 32^y = z^2$ has no positive integer solution. Sroysang [13] discussed the Diophantine equation $3^x + 5^y = z^2$.

Goel et al. [15] discussed the exponential Diophantine equation $M_5^p + M_7^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [16] proved that the exponential Diophantine equation $(2^{2m+1} - 1) + (6r+1)^n = ω^2$ has no solution in whole number. The exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ has studied by Kumar et al. [17]. Aggarwal and Sharma [18] studied the non-linear Diophantine equation $379^x + 397^y = z^2$ and proved that this equation has no solution in whole number.

The main aim of this article is to discuss the existence of solution of exponential Diophantine equation $439^p + 457^q = r^2$, where $p, q, r$ are whole numbers.

**Preliminaries:**

**Lemma 1** The exponential Diophantine equation $439^p + 1 = r^2$, where $p, r$ are whole numbers, has no solution in whole number.

**Proof:** Since 439 is an odd prime so $439^p$ is an odd number for all whole number $p$.

$⇒ 439^p + 1 = r^2$ is an even number for all whole number $p$.

$⇒ r$ is an even number.

$⇒ r^2 \equiv 0 \ (mod\ 3)$ or $r^2 \equiv 1 \ (mod\ 3)$ \hspace{1cm} (1)

Now, $439 \equiv 1 \ (mod\ 3)$

$⇒ 439^p \equiv 1 \ (mod\ 3)$, for all whole number $p$

$⇒ 439^p + 1 \equiv 2 \ (mod\ 3)$, for all whole number $p$

$⇒ r^2 \equiv 2 \ (mod\ 3)$ \hspace{1cm} (2)
Equation (2) contradicts equation (1). Hence exponential Diophantine equation $43^p + 1 = r^2$, where $p, r$ are whole numbers, has no solution in whole number.

**Lemma 2** The exponential Diophantine equation $457^q + 1 = r^2$, where $q, r$ are whole numbers, has no solution in whole number.

**Proof:** Since $457$ is an odd prime so $457^q$ is an odd number for all whole number $q$.

$\Rightarrow 457^q + 1 = r^2$ is an even number for all whole number $q$

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$  \hspace{1cm} (3)

Now, $457 \equiv 1(mod3)$

$\Rightarrow 457^q \equiv 1(mod3)$, for all whole number $q$

$\Rightarrow 457^q + 1 \equiv 2(mod3)$, for all whole number $q$

$\Rightarrow r^2 \equiv 2(mod3)$  \hspace{1cm} (4)

Equation (4) contradicts equation (3). Hence exponential Diophantine equation $457^q + 1 = r^2$, where $q, r$ are whole numbers, has no solution in whole number.

**Main Theorem:** The exponential Diophantine equation $43^p + 457^q = r^2$, where $p, q, r$ are whole numbers, has no solution in whole number.

**Proof:** There are four cases:

**Case: 1** If $p = 0$ then the exponential Diophantine equation $43^p + 457^q = r^2$ becomes

$1 + 457^q = r^2$, which has no solution in whole number by lemma 2.

**Case: 2** If $q = 0$ then the exponential Diophantine equation $43^p + 457^q = r^2$ becomes

$43^p + 1 = r^2$, which has no solution in whole number by lemma 1.

**Case: 3** If $p, q$ are positive integers, then $43^p, 457^q$ are odd numbers.

$\Rightarrow 43^p + 457^q = r^2$ is an even number

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$  \hspace{1cm} (5)

Now, $43 \equiv 1(mod3)$
\[ p \equiv 1 \pmod{3} \quad \text{and} \quad 457 \equiv 1 \pmod{3} \]
\[ 439^p \equiv 1 \pmod{3} \quad \text{and} \quad 457^q \equiv 1 \pmod{3} \]
\[ 439^p + 457^q \equiv 2 \pmod{3} \]
\[ r^2 \equiv 2 \pmod{3} \quad \text{(6)} \]

Equation (6) contradicts equation (5). Hence exponential Diophantine equation \( 439^p + 457^q = r^2 \) has no solution in whole number.

**Case:** 4 If \( p, q = 0 \), then \( 439^p + 457^q = 1 + 1 = 2 = r^2 \), which is impossible because \( r \) is a whole number. Hence exponential Diophantine equation \( 439^p + 457^q = r^2 \) has no solution in whole number.

**Conclusion:** In this article, authors successfully discussed the solution of exponential Diophantine equation \( 439^p + 457^q = r^2 \), where \( p, q, r \) are whole numbers and determined that the exponential Diophantine equation \( 439^p + 457^q = r^2 \), where \( p, q, r \) are whole numbers, has no solution in whole number.

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**References:**


