Stability and Convergence of Anomalous Diffusion Equation of Fractional Order

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Abstract: The aim of this paper is to develop the explicit finite difference scheme for time fractional anomalous diffusion equation. Furthermore we discuss the stability and convergence of the scheme.

***Index Terms* -** Fractional calculus, Finite difference, Caputo formula, Stability, Convergence.

I. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. In the recent scenario fractional calculus has many applications in physics, engineering, bio-science, applied mathematics, finance etc. [1,2,5,6]. In the framework of fractional calculus and applications anomalous diffusion equation has received great interest. A physical approach to anomalous diffusion equation containing fractional order derivatives in time or space or time-space [3,4,7,8,9,10,11]. As analytical solution of fractional diffusion equation is very difficult to find thus researchers develop the finite difference schemes to find numerical solution [12,13,14,16,17.18].

In this study we develop the time fractional explicit finite difference scheme for time fractional anomalous diffusion equation (TFADE). We consider the following [TFADE],

$$\frac{\partial^\alpha \mathrm{u(x,t)}}{\partial t^\alpha} = D\frac{\partial^2 \mathrm{u(x,t)}}{\partial t^2} + \lambda \mathrm{u(x,t)},\ 0 \le \alpha \le 1, (x,t) \in [0,L] \times [0,T] \tag{1.1}$$

initial condition: $u(x,0) = f(x), 0 \le x \le L$ (1.2)

boundary conditions: $u(0,t) = 0\ and\ \mathrm{u(L,t)} = 0, t \ge 0$ (1.3)

Definition 1.1:- The Caputo time-fractional derivative of order $\alpha, (0 < \alpha \le 1)$ is defined by,

$$\frac{\partial^\alpha \mathrm{u}(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,t)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}\ ; 0 < \alpha < 1$$
$$= \frac{\partial u(x,t)}{\partial \eta}\ ; \qquad \alpha = 1$$

We organize the paper as follows: In section 2, we develop explicit finite difference scheme for time fractional anomalous diffusion equation (TFADE). The section 3, is devoted for stability of the solution of the scheme and the convergence of the approximated finite difference scheme is proved in section 4.

II. FINITE DIFFERENCE SCHEME

In this section, we develop the explicit finite difference scheme for time fractional anomalous diffusion equation (TFADE) (1.3)-(1.5).

We define,

$$t_k = k\tau\ ;\ k = 0,1,2, \dots, N\ and\ x_i = ih\ ;\ i = 0,1,2, \dots, N$$

where

$$\tau = \frac{T}{N}\ \ and\ h = \frac{L}{M}$$

Let $u(x_{i,}t_k); i = 0,1,2, \dots, M\ and\ k = 0,1,2, \dots, N$ be the exact solution of (TFADE) (1.1)-(1.3) at mesh point $(x_{i,}t_k)$. Let u_i^k be the numerical approximation of the point $u(x_{i,}t_k)$. The time fractional derivative is approximated by the following scheme,

$$\frac{\partial^\alpha u(x_i,t_k)}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\eta}{(t_{k+1}-\eta)^\alpha}+O(\tau)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{(k-j)\tau}^{(k-j+1)\tau}\frac{d\xi}{\xi^\alpha}+O(\tau)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\left[\frac{(j+1)^{1-\alpha}-j^{1-\alpha}}{1-\alpha}\right]\tau^{1-\alpha}+O(\tau)$$
$$=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[u_i^{k+1}-u_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[u_i^{k-j+1}-u_i^{k-j}\right]+O(\tau)$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}, j=0,1,2,\dots,N$

Now for approximating second order space derivative, we adopt a symmetric second order difference quotient in space at time level $t=t_k$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u(x_{i-1},t_k)-2u(x_i,t_k)+u(x_{i+1},t_k)}{h^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i-1}^k-u_i^k+u_{i+1}^k}{h^2}$$

and

$$\frac{\partial u}{\partial x}=\frac{u_{i+1}^k-u_{i-1}^k}{2h}$$

Therefore substituting in equation (1.1), we get

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[u_i^{k+1}-u_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[u_i^{k-j+1}-u_i^{k-j}\right]=D\left[\frac{u_{i-1}^k-u_i^k+u_{i+1}^k}{h^2}\right]+\lambda\left[\frac{u_{i+1}^k-u_{i-1}^k}{2h}\right]$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}; j=0,1,2,\dots,k$

$$\left[u_i^{k+1}-u_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[u_i^{k-j+1}-u_i^{k-j}\right]=\frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}\left[u_{i-1}^k-u_i^k+u_{i+1}^k\right]+\frac{\lambda\Gamma(2-\alpha)\tau^\alpha}{2h}u_i^k$$

put $r=\frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$ and $\mu=\frac{\lambda\Gamma(2-\alpha)\tau^\alpha}{2h}$

we have,

$$\left[u_i^{k+1}-u_i^k\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j\left[u_i^{k-j+1}-u_i^{k-j}\right]=r\left[u_{i-1}^k-2u_i^{k+1}+u_{i+1}^k\right]+\mu\left[u_{i+1}^k-u_{i-1}^k\right] \quad (2.1)$$

After simplification, we get

$$u_i^{k+1}=(r-\mu)u_{i-1}^k+(1-2r-b_1)u_i^k+(r+\mu)u_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})u_i^{k-j}+b_ku_i^0 \quad (2.2)$$

where $r=\frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$; $\mu=\frac{\lambda\Gamma(2-\alpha)\tau^\alpha}{2h}$; $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}$; $j=0,1,2,\dots,k$;
$i=0,1,2,\dots,M$ and $k=0,1,2,\dots,N$.

The initial condition is approximated as $u_i^0=f(x_i), i=0,1,2,\dots,M$.The boundary conditions is approximated as $u_0^k=g_1(t_k), u_L^k=g_2(t_k), k=0,1,2,\dots,N$

Therefore, the complete fractional approximated initial boundary value problem is,

$$u_i^1=(r-\mu)u_{i-1}^0+(1-2r)u_i^0+(r+\mu)u_{i+1}^0; for\ k=0 \quad (2.3)$$
$$u_i^{k+1}=(r-\mu)u_{i-1}^k+(1-2r-b_1)u_i^k+(r+\mu)u_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})u_i^{k-j}+b_ku_i^0\ ; for\ k\geq 1 \quad (2.4)$$

initial condition: $u_i^0=f(x_i), i=0,1,2,\dots,M$ (2.5)

boundary conditions: $u_0^k=g_1(t_k), u_L^k=g_2(t_k), k=0,1,2,\dots,N$ (2.6)

where $r=\frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$; $\mu=\frac{\lambda\Gamma(2-\alpha)\tau^\alpha}{2h}$; $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}$; $j=0,1,2,\dots,k$;

$i = 0,1,2, \dots, M \quad and \quad k = 0,1,2, \dots, N$.

The problem (2.3)-(2.6) is a complete discretization of the problem (1.1)-(1.3).
Therefore, the fractional approximated initial boundary value problem (2.3)-(2.6) can be written in the following matrix equation form

$$U^1 = AU^0; for\ k = 0 \tag{2.7}$$

$$U^{k+1} = BU^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_k U^0; for\ k \geq 1 \tag{2.8}$$

where

$$A = \begin{pmatrix} 1-2r & r+\mu & \cdots & \cdots & \cdots \\ r-\mu & 1-2r & r+\mu & \cdots & \cdots \\ \vdots & \vdots & \ddots & r-\mu & 1-2r \end{pmatrix};$$

$$B = \begin{pmatrix} 1-2r-b_1 & r+\mu & \cdots & \cdots & \cdots \\ r-\mu & 1-2r-b_1 & r+\mu & \cdots & \cdots \\ \vdots & \vdots & \ddots & r-\mu & 1-2r-b_1 \end{pmatrix};$$

$U^k = \left[u_1^k, u_2^k, u_3^k, \dots \dots \dots, u_M^k\right]^T; k = 0,1,2, \dots, N$

$r = \dfrac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}; \mu = \dfrac{\lambda\Gamma(2-\alpha)\tau^\alpha}{2h}; b_j = (j+1)^{1-\alpha} - j^{1-\alpha};\ j = 0,1,2, \dots, k$;

$i = 0,1,2, \dots, M \quad and \quad k = 0,1,2, \dots, N$.

III. STABILITY

Lemma 3.1:- [14] The eigenvalues of the $N \times N$ tri-diagonal matrix

$$\begin{pmatrix} a & b & \cdot & & \cdot & \\ c & a & b & \cdot & \cdots & \cdot \\ \cdot & c & a & b & & \cdot \\ & \vdots & & \ddots & & \ddots \\ & & \cdots & c & a & b \\ & \cdot & & \cdot & c & a \end{pmatrix}$$

are given as

$$\lambda_s = a + 2\sqrt{bc}\ \cos\frac{s\pi}{N+1}; s = 1,2, \dots, N$$

where $a, b\ and\ c$ may be real or complex.

Theorem(3.1):-The solution of the fractional order explicit finite difference scheme (2.3)-(2.6) for the time fractional anomalous diffusion equation (1.1)-(1.3) is stable when

$$r \leq \left\{\frac{1+\mu^2}{2}, \frac{1}{2}(1-b_1) + \frac{\mu^2}{2(1-b_1)}\right\}$$

Proof :- We shall use the mathematical induction to analyze the stability. For $k = 0\ and\ 1 \leq i \leq M-1$ the eigenvalues of the matrix A are given by,

$$\lambda_s = 1 - 2r + 2\sqrt{(r+\mu)(r-\mu)}\ \cos\frac{s\pi}{M} \leq 1$$

and

$$\lambda_s = 1 - 2r + 2\sqrt{(r+\mu)(r-\mu)}\ \cos\frac{s\pi}{M}$$
$$\leq 1 - 2r - 2\sqrt{(r^2-\mu^2)}$$
$$\geq -1$$

$$\text{when } 1 - 2r + 2\sqrt{(r+\mu)(r-\mu)} \geq -1$$
$$2r + 2\sqrt{(r^2-\mu^2)} \leq 2$$
$$r + \sqrt{(r^2-\mu^2)} \leq 1$$
$$\sqrt{(r^2-\mu^2)} \leq 1 - r$$

$$r \le \frac{1+\mu^2}{2}$$

$\therefore \quad -1 \le \lambda_s \le 1; when\ \ r \le \frac{1+\mu^2}{2}$

$\therefore \quad |\lambda_s| \le 1; when\ \ r \le \frac{1+\mu^2}{2}$

Therefore $\quad ||A||_2 \le \max_{1\le i\le M-1}|\lambda_s| \le 1; when\ \ r \le \frac{1+\mu^2}{2}$

$\therefore\ ||A||_2 \le 1$

$$||U^1||_2 = ||AU^0||_2 \le ||A||_2||U^0||_2 \le ||U^0||_2$$

$\therefore\ ||U^1||_2 \le ||U^0||_2$

Thus the result is true for $n = 1$.

We assume that the result is true for $n = k$

i.e. $||U^k||_2 \le \left||U^0|\right|_2;\ when\ \ r \le \frac{1+\mu^2}{2}$

we prove that $||U^{k+1}||_2 \le \left||U^0|\right|_2$

for $||B||_2$, we have

$$\lambda_s = 1 - 2r - b_1 + 2\sqrt{(r+\mu)(r-\mu)}\ \ cos\frac{s\pi}{M} \le 1 - 2r - b_1 + 2\sqrt{(r^2-\mu^2)} \le 1 - b_1$$

also

$$\lambda_s = 1 - 2r - b_1 + 2\sqrt{(r+\mu)(r-\mu)}\ \ cos\frac{s\pi}{M}$$
$$\ge 1 - 2r - b_1 - 2\sqrt{(r+\mu)(r-\mu)}$$
$$\ge 1 - 2r - b_1 - 2\sqrt{(r^2-\mu^2)} \ge -(1-b_1)$$

when $\ 1 - 2r - b_1 - 2\sqrt{(r+\mu)(r-\mu)} \ge -(1-b_1)$

$$-2r - b_1 - 2\sqrt{(r^2-\mu^2)} \ge -2 + \ b_1$$
$$\sqrt{(r^2-\mu^2)} \le 1 - b_1 - r$$

$$r \le \frac{1}{2}(1-b_1) + \frac{1}{2}\frac{\mu^2}{(1-b_1)}$$

$\therefore \quad |\lambda_s| \le 1 - b_1; when\ \ r \le \frac{1}{2}(1-b_1) + \frac{1}{2}\frac{\mu^2}{(1-b_1)}$

$\therefore\ ||B||_2 \le 1 - b_1; when\ \ r \le \frac{1}{2}(1-b_1) + \frac{1}{2}\frac{\mu^2}{(1-b_1)}$

$$\therefore ||U^{k+1}||_2 = ||BU^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_kU^0||_2 \le (1 - b_1 + b_1 - b_k + b_k)||U^0||_2 \le ||U^0||_2$$

Hence, we prove that $||U^{k+1}||_2 \le \left||U^0|\right|_2; when\ \ r \le \frac{1}{2}(1-b_1) + \frac{1}{2}\frac{\mu^2}{(1-b_1)}$

Therefore, by mathematical induction

$$||U^k||_2 \le \left||U^0|\right|_2; when\ \ r \le min\left\{\frac{1+\mu^2}{2}, \frac{1}{2}(1-b_1) + \frac{\mu^2}{2(1-b_1)}\right\}$$

This proves that, the scheme is stable when

$$r \le min\left\{\frac{1+\mu^2}{2}, \frac{1}{2}(1-b_1) + \frac{\mu^2}{2(1-b_1)}\right\}$$

IV. CONVERGENCE

In this section, we discuss the convergence of the fractional order finite difference scheme (1.1) - (1.3).

Theorem 4.1 Let $\bar{U}^k$ be the vector of exact solution and U^k be the vector of approximate solution of the time fractional anomalous diffusion equation (1.1)-(1.3), then U^k converges to $\bar{U}^k$ as $(h,\tau) \to (0,0)$ when

$$r \le min\left\{\frac{1+\mu^2}{2}, \frac{1}{2}(1-b_1) + \frac{\mu^2}{2(1-b_1)}\right\}$$

Proof: Since $U^k = [u_1, u_2, \dots, u_{M-1}]^T$, $\bar{U}^k = [\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{M-1}]^T$ then $E^k = \bar{U}^k - U^k = [e_1^k, e_2^k, \dots, e_{M-1}^k]^T$.

Let us assume that, $|e_l^k| = \max_{1\le i\le M-1} |e_i^k| = ||E^\infty||_\infty; for\ l = 1,2,3, \dots.$

and $|T_l^k| = \max_{1\le i\le M-1} |T_i^k| = h^2 O(\tau^{1-\alpha} + h^2); for\ l = 1,2,3, \dots.$

For $k = 0$, from equation (2.3), we have

$$\begin{aligned} |e_l^k| &= |(r-\mu)e_{i-1}^0 + (1-2r)e_i^0 + (r+\mu)e_{i+1}^0| + r|T_i^1| \\ &\le |(r-\mu)e_{i-1}^0| + |(1-2r)e_i^0| + |(r+\mu)e_{i+1}^0| + r|T_i^1| \\ &\le |(r-\mu)e_l^0| + |(1-2r)e_l^0| + |(r+\mu)e_l^0| + r|T_l^1| \\ &\le (r-\mu+1-2r+r+\mu)|e_l^o| + r|T_l^1| \\ &\le |e_l^o| + D\tau^\alpha\Gamma(2-\alpha)O(\tau^{1-\alpha}+h^2) \end{aligned}$$

$||E^1||_\infty \le ||E^0||_\infty + D\tau^\alpha\Gamma(2-\alpha)O(\tau^{1-\alpha}+h^2)$

This proves that the result is true for n=1. Let us assume that the result is true for n=k that is

$||E^k||_\infty \le ||E^0||_\infty + D\tau^\alpha\Gamma(2-\alpha)O(\tau^{1-\alpha}+h^2)$

Now, we prove that the result is true for n=k+1, that is for this we show

$||E^{k+1}||_\infty \le ||E^0||_\infty + D\tau^\alpha\Gamma(2-\alpha)O(\tau^{1-\alpha}+h^2)$

From equation (2.4), we have

$$|e_l^{k+1}| = \left|(r-\mu)e_{i-1}^k + (1-2r-b_1)e_i^k + (r+\mu)e_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_k e_i^0\right| + r|T_i^k|$$

$$\le |(r-\mu)e_{i-1}^k| + |(1-2r-b_1)e_i^k| + |(r+\mu)e_{i+1}^k| + \left|\sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j}| + |b_k e_i^0\right|$$

$$+ r|T_i^k|$$

$$\le |(r-\mu)e_l^k| + |(1-2r-b_1)e_l^k| + |(r+\mu)e_l^k| + \left|\sum_{j=1}^{k-1}(b_j - b_{j+1})e_l^{k-j}| + |b_k e_l^0\right| + r|T_l^k|$$

$$\le |e_l^k| + r|T_l^k| \le |e_l^o| + r|T_l^k|$$

$\therefore \quad ||E^{k+1}||_\infty \le ||E^0||_\infty + D\tau^\alpha\Gamma(2-\alpha)O(\tau^{1-\alpha}+h^2)$

Thus, we have proved that, if we assume $r \le min\left\{\frac{1+\mu^2}{2}, \frac{1}{2}(1-b_1) + \frac{\mu^2}{2(1-b_1)}\right\}$, then $||E^k||_\infty \to 0\ as\ (h,\tau) \to (0,0)$, which results in the convergence of U^k to $\bar{U}^k$.

V.CONCLUSION

We develop the explicit finite difference scheme for time fractional anomalous diffusion equation (TFADE). Furthermore we discuss its stability and convergence of the scheme.

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