# Some distance-based topological indices of the strong product of a graph and a diameter two graph

R. Lakshmi[1] and O. Lakshmi[2]
Department of Mathematics, Annamalai University,
Annamalainagar 608 002, India.

**Abstract**

The *strong product* $G \boxtimes H$ of simple graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and edge set $\{(u,x)(v,y) | uv \in E(G)$ and $xy \in E(H)$, or $uv \in E(G)$ and $x = y$, or $u = v$ and $xy \in E(H)\}$.

For a connected graph $G$, the *Wiener index* $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$, the *hyper-Wiener index* $WW(G) = \frac{1}{2}W(G) + \frac{1}{4}\sum_{u,v\in V(G)} (d_G(u,v))^2$, and the *Harary index* $H(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{1}{d_G(u,v)}$.

In this paper, for a connected graph $G$, we calculate the exact values of Wiener index, hyper-Wiener index and Harary index of $G \boxtimes H_0$, where $H_0$ is a graph of diameter 2.

*Keywords:* Wiener index, hyper-Wiener index, Harary index, Strong product, Diameter.
sMathematics Subject Classification (2010): Primary: 05C12, 05C76; Secondary: 05C90.

## 1 Introduction

All graphs considered in this paper are finite, simple, undirected, connected and with at least two vertices. Let $G = (V(G), E(G))$ be a connected graph of *order* $n(G) = |V(G)|$ and *size* $m(G) = |E(G)|$. For vertices $u, v \in V(G)$, the *distance* between $u$ and $v$ in $G$, denoted by $d_G(u,v)$, is the length of a shortest $(u,v)$-path in $G$. For $v \in V(G)$, $d_G(v)$ is the *degree* of the vertex $v$. The *diameter* of $G$, denoted by diam$(G)$, is $\max\{d_G(u,v): u, v \in V\}$.

A *topological index* of a graph is a parameter related to the graph, it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacological, toxicological, biological and other properties of chemical compounds [4]. A topological index related to distance is called a ``distance-based topological index". Several types of such indices exist, especially those based on vertex and edge distances. One of the oldest and most intensively studied topological indices is the Wiener index (see [14, 13]). Its chemical applications and mathematical properties are well studied in [3, 8].

For a connected graph $G$, the *Wiener index* of $G$ is

$$W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v) = \frac{1}{2}\sum_{u\in V(G)} D_G(u),$$

where

$$D_G(u) = \sum_{v\in V(G)} d_G(u,v).$$

The *hyper-Wiener index* of $G$ is

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{4}\sum_{u,v\in V(G)} (d_G(u,v))^2.$$

The hyper-Wiener index of an acyclic graph was first introduced by Randic [12]. Applications of the hyper-Wiener index as well as its calculation are well studied in [6, 7].

The *Harary index*

$$H(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{1}{d_G(u,v)}$$

of $G$ has been introduced by Plavsic *et al*. [11] and independently by Ivanciuc *et al*. [5]. Its applications and mathematical properties are well studied in [2].

[1]e-mail: mathlakshmi@gmail.com.
[2]e-mail: lakshmiosuranv@gmail.com.

Let $G$ and $H$ be simple graphs. The *strong product* of $G$ and $H$, denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$ and edge set

$$E(G \boxtimes H) = \{(u,x)(v,y) | uv \in E(G) \text{ and } xy \in E(H), \text{ or } uv \in E(G) \text{ and } x = y, \text{ or } u = v \text{ and } xy \in E(H)\}.$$

Notations and terminology not defined here can be found in [1].

In this paper, we compute $W(H_0)$, $WW(H_0)$, $H(H_0)$, $W(G \boxtimes H_0)$, $WW(G \boxtimes H_0)$, and $H(G \boxtimes H_0)$, where $G$ is a connected graph, and $H_0$ is a graph of diameter 2.

## 2 Main Result

**Theorem 2.1** *Let* $G$ *be a connected graph and let* $H_0$ *be a graph of diameter* 2. *Then*

1. $W(H_0) = n(H_0)^2 - n(H_0) - m(H_0)$,

2. $WW(H_0) = \frac{3}{2}n(H_0)^2 - \frac{3}{2}n(H_0) - 2m(H_0)$,

3. $H(H_0) = \frac{1}{4}n(H_0)^2 - \frac{1}{4}n(H_0) + \frac{1}{2}m(H_0)$,

4. $W(G \boxtimes H_0) = (W(G) + n(G) + m(G))\ n(H_0)^2 - (n(G) + m(G))n(H_0) - (n(G) + 2m(G))m(H_0)$,

5. $WW(G \boxtimes H_0) = (WW(G) + \frac{3}{2}n(G) + 2m(G))n(H_0)^2 - (\frac{3}{2}n(G) + 2m(G))n(H_0) - 2(n(G) + 2m(G))m(H_0)$,

6. $H(G \boxtimes H_0) = (H(G) + \frac{1}{4}n(G) - \frac{1}{2}m(G))n(H_0)^2 - (\frac{1}{4}n(G) - \frac{1}{2}m(G))n(H_0) + (\frac{1}{2}n(G) + m(G))m(H_0)$.

*Proof of* 1. As $H_0$ is of diameter 2,

$$W(H_0) = \frac{1}{2}\sum_{x,y \in V(H_0)} d_{H_0}(x,y)$$
$$= \frac{1}{2}([2m(H_0) \times 1] + [2(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)) \times 2])$$
$$= n(H_0)^2 - n(H_0) - m(H_0).$$

*Proof of* 2.

$$\sum_{x,y \in V(H_0)} (d_{H_0}(x,y))^2 = [2m(H_0) \times 1^2] + \left[2\left(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)\right) \times\right.$$

$2^2]$

$$= 4n(H_0)^2 - 4n(H_0) - 6m(H_0). \qquad (1)$$

Hence,

$$WW(H_0) = \frac{1}{2}W(H_0) + \frac{1}{4}\sum_{x,y\in V(H_0)} (d_{H_0}(x,y))^2$$
$$= \frac{1}{2}(n(H_0)^2 - n(H_0) - m(H_0)) + \frac{1}{4}(4n(H_0)^2 - 4n(H_0) - 6m(H_0))$$
$$= \frac{3}{2}n(H_0)^2 - \frac{3}{2}n(H_0) - 2m(H_0).$$

*Proof of* 3.

$$H(H_0) = \frac{1}{2}\sum_{x,y\ \in\ V(H_0)} \frac{1}{d_{H_0}(x,y)}$$
$$= \frac{1}{2}([2m(H_0) \times \frac{1}{1}] + [2(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)) \times \frac{1}{2}])$$
$$= \frac{1}{4}n(H_0)^2 - \frac{1}{4}n(H_0) + \frac{1}{2}m(H_0). \qquad (2)$$

*Proof of* 4. By definition,

$$2W(G \boxtimes H_0) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(v,y)).$$

- Terms with $u = v$ contribute

$$\sum_{(u,x),(u,y)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(u,y)) = \sum_{u\in V(G);x,y\in V(H_0)} d_{H_0}(x,y)$$
$$= n(G)\sum_{x,y\in V(H_0)} d_{H_0}(x,y) = n(G)(2W(H_0))$$
$$= 2n(G)(n(H_0)^2 - n(H_0) - m(H_0)).$$

- Terms with $x = y$ contribute

$$\sum_{(u,x),(v,x)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(v,x)) = \sum_{u,v\in V(G);x\in V(H_0)} d_G(u,v)$$
$$= n(H_0)\sum_{u,v\in V(G)} d_G(u,v) = n(H_0)(2W(G)) = 2W(G)n(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \in E(H_0)$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(v,y)) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} 1$$
$$= 4m(G)m(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \notin E(H_0)$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(v,y)) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} 2$$
$$= 4m(G)(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)) \times 2$$
$$= 4m(G)(n(H_0)^2 - n(H_0)) - 8m(G)m(H_0).$$

- Terms with $u \neq v, uv \notin E(G), x \neq y$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} d_{G\boxtimes H_0}((u,x),(v,y)) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} d_G(u,v)$$
$$= n(H_0)(n(H_0)-1)\sum_{uv\notin E(G)} d_G(u,v)$$
$$= (n(H_0)^2 - n(H_0))(2W(G) - 2m(G))$$
$$= 2(W(G) - m(G))(n(H_0)^2 - n(H_0)).$$

The expression for $W(G \boxtimes H_0)$ follows by adding the above five contributions, simplifying it, and dividing by 2.

*Proof of* 5. By definition,

$$4WW(G \boxtimes H_0) - 2W(G \boxtimes H_0) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(v,y)))^2.$$

- Terms with $u = v$ contribute

$$\sum_{(u,x),(u,y)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(u,y)))^2 = \sum_{u\in V(G);x,y\in V(H_0)} (d_{H_0}(x,y))^2$$
$$= n(G)\sum_{x,y\in V(H_0)} (d_{H_0}(x,y))^2 = n(G)[4n(H_0)^2 - 4n(H_0) -$$

$6m(H_0)]$ (by (1))

$$= 2n(G)(2n(H_0)^2 - 2n(H_0) - 3m(H_0)).$$

- Terms with $x = y$ contribute

$$\sum_{(u,x),(v,x)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(v,x)))^2 = \sum_{u,v\in V(G);x\in V(H_0)} (d_G(u,v))^2$$
$$= n(H_0)\sum_{u,v\in V(G)} (d_G(u,v))^2 = n(H_0)(4WW(G) - 2W(G))$$

$$= 2(2WW(G) - W(G))n(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \in E(H_0)$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(v,y)))^2 = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} 1^2$$
$$= 4m(G)m(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \notin E(H_0)$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(v,y)))^2 = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} 2^2$$
$$= 4m(G)(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)) \times 4$$
$$= 8m(G)(n(H_0)^2 - n(H_0)) - 16m(G)m(H_0).$$

- Terms with $u \neq v, uv \notin E(G), x \neq y$, contribute

$$\sum_{(u,x),(v,y)\in V(G)\times V(H_0)} (d_{G\boxtimes H_0}((u,x),(v,y)))^2 = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} (d_G(u,v))^2$$
$$= n(H_0)(n(H_0)-1)\sum_{uv\notin E(G)} (d_G(u,v))^2$$
$$= n(H_0)(n(H_0)-1)(\sum_{u,v\in V(G)} (d_G(u,v))^2 - \sum_{uv\in E(G)} (d_G(u,v))^2)$$
$$= (n(H_0)^2 - n(H_0))\left(4WW(G) - 2W(G) - \sum_{uv\in E(G)} 1^2\right)$$
$$= (n(H_0)^2 - n(H_0))(4WW(G) - 2W(G) - 2m(G))$$
$$= 2(2WW(G) - W(G) - m(G))(n(H_0)^2 - n(H_0)).$$

Adding the above five contributions, we get

$$4WW(G\boxtimes H_0) - 2W(G\boxtimes H_0)$$
$$= 2(2WW(G) - W(G) + 2n(G) + 3m(G))n(H_0)^2$$
$$-2(2n(G) + 3m(G))n(H_0) - 6(n(G) + 2m(G))m(H_0).$$

Hence,

$$2WW(G\boxtimes H_0)$$
$$= W(G\boxtimes H_0) + (2WW(G) - W(G) + 2n(G) + 3m(G))n(H_0)^2$$
$$-(2n(G) + 3m(G))n(H_0) - 3(n(G) + 2m(G))m(H_0)$$
$$= (2WW(G) + 3n(G) + 4m(G))n(H_0)^2$$
$$-(3n(G) + 4m(G))n(H_0) - 4(n(G) + 2m(G))m(H_0).$$

To obtain the expression for $WW(G\boxtimes H_0)$, divide by $2$.

*Proof of* 6. By definition,

$$2H(G\boxtimes H_0) = \sum_{(u,x),(v,y)\in V(G)\times V(H_0)} \frac{1}{d_{G\boxtimes H_0}((u,x),(v,y))}.$$

- Terms with $u = v$ contribute

$$\sum_{(u,x),(u,y)\in V(G)\times V(H_0)} \frac{1}{d_{G\boxtimes H_0}((u,x),(u,y))} = \sum_{u\in V(G);\ x,y\in V(H_0)} \frac{1}{d_{H_0}(x,y)}$$
$$= n(G)\sum_{x,y\in V(H_0)} \frac{1}{d_{H_0}(x,y)} = n(G)(\tfrac{1}{2}n(H_0)^2 - \tfrac{1}{2}n(H_0) + m(H_0)).\ \text{(by (2))}$$

- Terms with $x = y$ contribute

$$\sum_{(u,x),(v,x)\in V(G)\times V(H_0)} \frac{1}{d_{G\boxtimes H_0}((u,x),(v,x))} = \sum_{u,v\in V(G);\ x\in V(H_0)} \frac{1}{d_G(u,v)}$$

$$= n(H_0) \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)} = n(H_0)(2H(G)) = 2H(G)n(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \in E(H_0)$, contribute

$$\sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{d_{G \boxtimes H_0}((u,x),(v,y))} = \sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{1}$$
$$= 4m(G)m(H_0).$$

- Terms with $u \neq v, uv \in E(G), x \neq y, xy \notin E(H_0)$, contribute

$$\sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{d_{G \boxtimes H_0}((u,x),(v,y))} = \sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{2}$$
$$= 4m(G)(\frac{n(H_0)(n(H_0)-1)}{2} - m(H_0)) \times \frac{1}{2}$$
$$= m(G)(n(H_0)^2 - n(H_0) - 2m(H_0)).$$

- Terms with $u \neq v, uv \notin E(G), x \neq y$, contribute

$$\sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{d_{G \boxtimes H_0}((u,x),(v,y))} = \sum_{(u,x),(v,y) \in V(G) \times V(H_0)} \frac{1}{d_G(u,v)}$$
$$= n(H_0)(n(H_0)-1) \sum_{uv \notin E(G)} \frac{1}{d_G(u,v)}$$
$$= (n(H_0)^2 - n(H_0))(2H(G) - 2m(G)).$$

Hence,

$$2H(G \boxtimes H_0) = (2H(G) + \frac{1}{2}n(G) - m(G))n(H_0)^2$$
$$-(\frac{1}{2}n(G) - m(G))n(H_0) + (n(G) + 2m(G))m(H_0).$$

To obtain the expression for $H(G \boxtimes H_0)$, divide by $2$.

## 3 Corollaries

In Theorem 2.1 (4), (5) and (6), when $H_0 = K_{m_0,m_1,\dots,m_{r-1}}$, the complete $r$-partite graph with partite sizes $m_0, m_1, \dots, m_{r-1}$, we have the following corollaries. Let $n_0 = \sum_{i=0}^{r-1} m_i$, the number of vertices of $K_{m_0,m_1,\dots,m_{r-1}}$ and let $q = |E(K_{m_0,m_1,\dots,m_{r-1}})|$.

**Corollary 3.1** *([9]). For any connected graph $G$,*

$$W(G \boxtimes K_{m_0,m_1,\dots,m_{r-1}}) = n_0^2 W(G) + n(G)(n_0^2 - q - n_0) + (n_0^2 - 2q - n_0)m(G).$$

**Corollary 3.2** *([9]). For any connected graph $G$,*

$$WW(G \boxtimes K_{m_0,m_1,\dots,m_{r-1}}) = n_0^2 WW(G) + \frac{n(G)}{2}(3n_0^2 - 4q - 3n_0) + 2(n_0^2 - 2q - n_0)m(G).$$

**Corollary 3.3** *([10]). For any connected graph $G$,*

$$H(G \boxtimes K_{m_0,m_1,\dots,m_{r-1}}) = n_0^2 H(G) - \frac{m(G)}{2}(n_0^2 - 2q - n_0) + \frac{n(G)}{4}(n_0^2 + 2q - n_0).$$

## References

[1] R. Balakrishnan and K. Ranganathan, A *Textbook of Graph Theory,* Second Edition, Springer, New York, 2012.

[2] K.C. Das, B. Zhou and N. Trinajstic, Bounds on Harary index, *J. Math. Chem.* 46 (2009) 1377-1393.

[3] A.A. Dobrynin, R. Entringer and I. Gutman, Wiener index of trees: Theory and applications, *Acta Appl. Math.* 66 (2001) 211-249.

[4] I. Gutman and O.E. Polansky, Mathematical concepts in organic chemistry, *Springer-Verlag,* Berlin, 1986.

[5] O. Ivancicuc, T.S. Balaban and A.T. Balaban, Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.* 12 (1993) 309-318.

[6] S. Klavzar, P. Zigert and I. Gutman, An algorithm for the calculation of the hyper-Wiener index of benzenoid hydrocarbons, *Comput. Chem.* 24 (2000) 229-233.

[7] I. Lukovits, A note on a formula for the hyper-Wiener index of some trees, *J. Chem. Inf. Comput. Sci.* 34 (1994) 1079-1081.

[8] D.E. Needham, I.C. Wei and P.G. Seybold, Molecular modeling of the physical properties of alkanes, *J. Amer. Chem. Soc.* 110 (1988) 4186-4194.

[9] K. Pattabiraman and P. Paulraja, Wiener and vertex PI indices of the strong product of graphs, *Discussiones Mathematicae Graph Theory,* 32 (2012) 749 -769.

[10] K. Pattabiraman and P. Paulraja, Harary index of product graphs, *Discussiones Mathematicae Graph Theory,* 35 (2015) 17-33.

[11] D. Plavsic , S. Nikolic, N. Trinajstic and Z. Mihalic, On the Harary index for the characterization of chemical graphs, *J. Math. Chem.* 12 (1993) 235-250.

[12] M. Randic, Novel molecular descriptor for structure-property studies, *Chem. Phys. Lett.* 211 (1993) 478-483.

[13] R. Todeschini and V. Consonni, Handbook of molecular descriptors, *Wiley-VCH,* Weinheim, 2000.

[14] H. Wiener, Structural determination of the paraffin boiling points, *J. Am. Chem. Soc.* 69 (1947) 17-20.