G-Inverse of Lower Triangular Block Operator Matrix

USHA S, and SENTHILKUMAR D
ASSISTANT PROFESSOR, PROFESSOR,
DEPARTMENT OF MATHEMATICS,
SRI SHAKTHI INSTITUTE OF ENGINEERING AND TECHNOLOGY, COIMBATORE, INDIA

Abstract: Inside this paper, we probe the depictions of Drazin spectrum \( \sigma_d(M_C) \) and Generalized inverse and generalized Drazin inverse of lower triangular operator matrix on Banach space.

Keywords: Operator Matrices, Drazin spectrum, single-valued extension property, Generalized inverse, Drazin inverse.

I. INTRODUCTION

An operator \( T_1 \in L(X) \) is said to be a Drazin invertible if there exists a positive integer \( k \) and an operator \( S_1 \in L(X) \) such that \( T_1^kS_1T_1 = T_1^k \), \( S_1T_1S_1 = S_1 \) and \( T_1S_1 = S_1T_1 \). The Drazin spectrum is defined by \( \sigma_D(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible} \} \).

The Drazin invertible spectrum is defined by \( \sigma_d(T_1) = \{ \lambda \in \mathbb{C} : T_1 - \lambda I \text{ is not Drazin invertible operator} \} \).

The Drazin invertible operator is defined by an operator \( T_1 \in L(X) \) is said to be Drazin invertible if \( T_1 \) is both left and Right Drazin invertible.

It is well known that \( T \) is Drazin invertible if and only if \( T \) is of finite ascent and descent, which is also equivalent to the fact that \( T = R \oplus N \) where \( R \) is invertible and \( N \) nilpotent (see [16, Corollary 2.2]). Clearly, \( T_1 \) is Drazin invertible if and only if \( T_1^* \) is Drazin invertible. A bounded linear operator \( T_1 \in L(X) \) is said to have the single-valued extension property (SVEP, for short) at \( \lambda \in \mathbb{C} \) if for every open neighborhood \( \cup \) of \( \lambda \), the constant function \( f \equiv 0 \) is the only analytic solution of the equation \( (T_1 - \mu)f(\mu) = 0 \) for all \( \mu \in \cup \).

We use \( S_1(T_1) \) to denote the open set where \( T_1 \) fails to have the SVEP and we say that \( T_1 \) has the SVEP if \( S_1(T_1) \) is the empty set, [12]. It is easy to see that \( (T_1) \) has the SVEP at every point \( \lambda \in \sigma(T) \), where \( \sigma(T) \) denotes the set of all isolated points of \( \sigma(T) \). Note that (see [12])

\[
S_1(T_1) \subseteq \sigma_p(T_1) \quad \text{and} \quad \sigma(T_1) = S_1(T_1) \cup \sigma_s(T_1)
\]

Also, it follows from [15] if \( T \) is of finite ascent and descent then \( T_1 \) and have the SVEP. Hence \( S_1(T_1) \cup S_1(T_1^*) \subseteq \sigma_d(T_1) \).
For $\mathcal{T}_1 \in L(X), \mathcal{T}_2 \in L(Y)$ and $C \in L(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$

$$by \quad M_C = \left[ \begin{array}{cc} \mathcal{T}_1 & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 \end{array} \right] \right].$$

In [11] it is proved that $\sigma(M_C) \cup [S_1(\mathcal{T}_1^*) \cap S_1(\mathcal{T}_2)] = \sigma(\mathcal{T}_1) \cup \sigma(\mathcal{T}_2)$. Numerous mathematicians were interested by the defect set $[\sigma(\mathcal{T}_1) \cup \sigma(\mathcal{T}_2)] \setminus \sigma(M_C)$.

See for instance [11, 13, 14] for the spectrum and the essential spectrum, [19] for the Weyl spectrum, [10] for the Browder spectrum and [9, 10] for the essential approximate point spectrum and the Browder essential approximate point spectrum. See also the references therein. For the Drazin spectrum, Campbell and Meyer [7] were the first studied the Drazin invertibility of $2 \times 2$ lower triangular operator matrices $M_C$ where $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are n x n complex matrices. They proved that $\sigma_d(M_C) \subseteq \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$.

D. S. Djordjević and P. S. Stanimirović generalized the inclusion (1.3) to arbitrary Banach spaces [8].

Inclusion (1.3) may be strict.

The generalised inverse (for short G-Inverse) and generalised Drazin inverse (for short GD-Inverse). Presume $T_n$ is a given lower triangular block matrix and $X_n$ is an arbitrary upper triangular block matrix. The generalised Drazin inverse of a $2 \times 2$ block operator matrix

$T = \left( \begin{array}{cc} \mathcal{T}_1 & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 \end{array} \right)$. Let $X$ and $K$ be separable, infinite dimensional, complex Banach spaces. Denote by $B(X, K)$ the set of all bounded linear operators from $X$ into $K$. For an operator $T \in B(X, K), R(A), N(A)$ denote the range, the null space and the adjoint of $A$, respectively. For $T \in B(X, K)$, if there exists $T^+ \in B(X, K)$ satisfying the following four operator equation,

$T^+ T = T, T^+ T^+ = T^+, T^+ T = (T^+ T)^*, T^+ T = (T^+ T)^*$, then $T^+$ is called the G-Inverse of $T$. It is well known that has the G-inverse if and only if $R(T)$ is closed and the G-inverse of $T$ is unique (see [16, 20, 24]).

1. Main results and its proof

**Theorem 1.1**

For $\mathcal{T}_1 \in L(X), \mathcal{T}_2 \in L(Y)$, and $\mathcal{T}_3 \in L(Y, X)$ we have

$$\sigma_d(M_C) \cup [S_1(\mathcal{T}_1^*) \cap S_1(\mathcal{T}_2)] = \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2).$$

**Proof**

Since the inclusion $\sigma_d(M_C) \cup [S_1(\mathcal{T}_1^*) \cap S_1(\mathcal{T}_2)] \subseteq \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$ always holds, it suffices to prove the reverse inclusion. Let $\lambda \in \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$, without loss of generality, we can assume that $\lambda = 0$. Then $M_C$ is of finite ascent and descent. Hence from [9, Lemma 2.1] we have $A$ is of finite ascent and $B$ is of finite descent. Also, by duality $\mathcal{T}_1^*$ is of finite descent and $\mathcal{T}_2^*$ is of finite ascent. For the sake of contradiction assume that

$$0 \notin S_1(\mathcal{T}_1^*) \cap S_1(\mathcal{T}_2).$$

Case 1. $0 \notin S_1(\mathcal{T}_1^*)$. Since $M_C$ is Drazin invertible, then there exists $\varepsilon > 0$ such that for every $\lambda$

$$0 < |\lambda| < \varepsilon, M_C - \lambda I \text{ is invertible. Hence } \mathcal{T}_1 - \lambda I \text{ is right invertible. Thus } 0 \notin acc\sigma_{ap}(\mathcal{T}_1) = acc\sigma_s(\mathcal{T}_1^*).$$

If $\mathcal{T}_1^*$ then $\mathcal{T}_1^*$ is Drazin invertible and so $\mathcal{T}_1$ is. Now if $0 \notin \sigma(\mathcal{T}_1^*)$, since $\sigma(\mathcal{T}_1^*) = S_1(\mathcal{T}_1^*) \cup S_1(\mathcal{T}_1^*)$ Then

$$0 \text{ is an isolated point of } \sigma(\mathcal{T}_1^*).$$

Now $\mathcal{T}_1^*$ is of finite decent and $0 \notin isos\sigma(\mathcal{T}_1^*)$. Hence it follows from [18, Theorem 10.5]

$\mathcal{T}_1^*$ is Drazin invertible. Thus $\mathcal{T}_1$ is Drazin invertible. Since $M_C$ is Drazin invertible it follows from [21, lemma 2.7] that $\mathcal{T}_2$ is also Drazin invertible which contradicts our assumption.

Case 2. $0 \notin S_1(\mathcal{T}_2^*)$, the proof goes similarly.
Theorem 1.2
Let \( T_1 \in B(X), T_2 \in B(K), T_3 \in B(K,X) \) and \( T_2 \) be invertible. Then 2 by 2 block operator valued matrix
\[
T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}
\]
is \( G \)-invertible if and only if \( R(T_1) \) is closed and
\[
\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix} = \begin{bmatrix} I & T_2 \Delta T_2^* (I - T_1 T_2^+) \\ I & -T_2 \Delta T_2^* (I - T_1 T_2^+) \end{bmatrix} \Delta T_3^*
\]
Proof
Since
\[
\begin{bmatrix} T_1^* & T_2^* \\ 0 & T_3^* \end{bmatrix} \begin{bmatrix} I & T_2^* (T_1 T_3^*)^{-1} \\ 0 & -(T_3^*)^{-1} \end{bmatrix} = \begin{bmatrix} T_1^* & 0 \\ 0 & I \end{bmatrix}
\]
\( R(T^*) \) is closed if and only if \( R(T_1^*) \). This shows that \( T \) is invertible if and only if \( R(T) \) is closed.
In this case \( T \) has the form
\[
\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & N(T_1) \\ 0 & 0 & 0 \\ T_{11} & T_{21} & T_3 \end{bmatrix} \begin{bmatrix} N(T_1) \\ 0 \\ K \end{bmatrix} \rightarrow \begin{bmatrix} N(T_1^*) \\ 0 \\ K \end{bmatrix}
\]
Where \( T_{11} \) as an operator from \( R(T_1^*) \) on to \( R(T) \) is invertible. Now \( N = \begin{bmatrix} 0 \\ T_{22} \end{bmatrix} \),
\( M = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_3 \end{bmatrix} \)
and
\[
\Delta = (T_2^* T_2 + T_3^* (I - T_1 T_3^*)) T_3^{-1} = (T_2^* T_2 + T_3^* T_3)^{-1}
\]
It is easy to check that
\[
\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}^+ = \begin{bmatrix} 0 & N^* \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix}^{-1}
\]
\[
= \begin{bmatrix} 0 & (N^* N + M^* M)^{-1} N^* \\ 0 & (N^* N + M^* M)^{-1} M^* \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & T_1 T_2^* \Delta T_2^* \\ I & \Delta T_3^* \end{bmatrix} \begin{bmatrix} T_1^* - T_1 T_2^* T_3^* (I - T_1 T_3^+) \\ \Delta T_2^* (I - T_1 T_2^+) \end{bmatrix}
\]
Remark
In Theorem 1, if \( R(T_2) \) is closed, we can show that \( T \) is \( G \)-invertible if and only if
\( R((I - T_2^* T_2) T_3) (I - T_2^* T_2) \) is closed in a similar way. In this case, \( T^+ \) has a very complicated representation. But we can show that \( T^+ \) has the form as
\[
\begin{bmatrix} T_1 & T_3 \\ 0 & T_2 \end{bmatrix}^{(1)} = \begin{bmatrix} T_1^+ - T_1^+ T_3 T_3^+ & -T_1^+ T_3 T_2^+ \\ T_3^+ & T_2^+ - T_3^+ T_3 T_2^+ \end{bmatrix}
\]
Were \( T_3 = ((I - T_1 T_1^*) T_3 (I - T_2^* T_2)) \).
In addition, if \( T_1^+ T_2^* T_3^+ (I - T_2^* T_2) T_3^+ = 0 \) and \( (I - T_2^* T_2 - T_3^+ T_3) T_3^+ T_2^+ = 0 \), a directly calculation can show that,
\[
\begin{bmatrix} T_1 & T_3 \\ 0 & T_2 \end{bmatrix}^{+} = \begin{bmatrix} T_1^+ - T_1^+ T_3 T_3^+ & -T_1^+ T_3 T_2^+ + T_1^+ T_3 T_3^+ T_3 T_2^+ \\ T_3^+ & T_2^+ - T_3^+ T_3 T_2^+ \end{bmatrix}
\]
(2) If we assume as well that \( R(T_3) \subset R(T_1) \) and \( R(T_3^+) \subset R(T_2^+) \), then \( T \) satisfies remark (1), and \( T_3 = (I - T_1 T_1^*) T_3 (I - T_2^* T_2) = 0 \). Then we have

References