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An Introduction of Laplace Transform for Engineering

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Abstract: Laplace Transformation has vital significance in engineering. Laplace Transform gives solution of many electrical engineering, mechanical engineering and communication engineering problems in easiest way. The Title of the paper is an introduction of Laplace Transform for engineering. This paper especially designed for second year engineering students. This paper deals with a brief introduction of what Laplace transform is and its properties and application in the applied science & Engineering problems. This article will provide basic fundamental clearance about Laplace Transform & gain some of the very important and basic applications of this transforms. Laplace transform is broadly used to solve ordinary differential equations, particularly initial value problems. In this article the main focus given to discuss in detail the definition of Laplace Transform, its formulae and properties with proof. It also include definition, formulae and properties of inverse Laplace Transform with examples then how apply Convolution theorem to find inverse Laplace Transform. Then focus on the formation of some special functions like periodic function, Heaviside's Unit –step function and Dirac-delta function or Unit impulse function. As well as we will find the solution of

some ordinary differential equations using Laplace Transform. By applying Laplace Transform we can convert ordinary differential equation into an algebraic equation.

Keywords: Laplace Transform, Convolution theorem, Heaviside's Unit –step function and Dirac-delta function, Unit impulse function

Introduction: In mathematics, the Laplace Transform named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of real variable 't' into a function of complex variable 's'. Laplace Transform has many applications in science and engineering because it is a tool for solving differential equations. Laplace Transform plays a vital role in control system engineering. Properties of both Laplace and Inverse Laplace Transformation are used to analyzing the dynamic control system. The Laplace Transformation method is used particularly effective in solving linear differential equations-ordinary as well as partial. It is generally used electrical circuit and systems problems. Laplace Transformation gives best where knowledge of the system transfer function is important such that in control theory, population growth and decay problems. By using the Laplace transform convert an ordinary differential equation into an algebraic equation, an algebraic equation easy to solve. Assume that the function f(t) is piecewise continuous function. A function is said to be piecewise continuous function if it has a finite number of breaks and it does not blow up to infinitely anywhere. Laplace Transform will be denoted by L{f(t) } or $\overline{f}(s)$ where f(t) is a function of 't'.

I. Laplace Transform

Definition-

Let f(t) be a function of t defined for all t ≥ 0. Then the Laplace Transform of f(t), denoted by L{f(t)}and it is defined by,

$$L\{f(t)\}= \int_0^\infty e^{-st}\, f(t)\, dt \qquad \ldots(I)$$

Provided that the integral exist, 's' is a parameter which may be real or complex. L{f(t)} is clearly a function of s and is briefly written as $\overline{f}(s)$ that is L{f(t)}= $\overline{f}(s)$

Example: Find Laplace Transform of f(t), where f(t)=3 ; $0< t < 5$ and f(t)=0 ; t > 5

Solution: By the definition of Laplace Transform

$$L\{f(t)\}= \int_0^\infty e^{-st} f(t)\, dt = \int_0^5 e^{-st} f(t)\, dt + \int_5^\infty e^{-st} f(t)\, dt= \int_0^5 e^{-st} (5)\, dt + \int_5^\infty e^{-st}(0)\, dt$$

$$= 5 \int_0^5 e^{-st}\, dt + 0= 5[\frac{e^{-st}}{(-s)}]_0^5 = 5[\frac{e^{-5s}}{(-s)} - \frac{e^{-0s}}{(-s)}] = 5[\frac{e^{-5s}}{(-s)} + \frac{1}{s}]$$

$$L\{f(t)\}= \frac{5}{s}\ (1 - e^{-5s})$$

Linearity Property- If k_1 and k_2 are two constants then,

$$L\{ k_1 f_1(t) + k_2 f_2(t) \} = k_1 L\{f_1(t)\}+ k_2\, L\{f_2(t)\}$$

This means the Laplace Transform of sum of two functions is equal to the sum of their Laplace Transforms.

For example, L{ $t^2 + 4\boldsymbol{e^{3t}} - \cos 2t$}= L{$t^2$}+ 4L{$\boldsymbol{e^{3t}}$}- L{$\cos 2t$}= $\frac{2!}{s^3} + \frac{4}{s-3}$ - $\frac{s}{s^2+2^2}$

Laplace Transform of standard functions:

1) L{e^{at}} = $\frac{1}{s-a}$

Proof- By the definition of Laplace Transform

$$L\{f(t)\}= \int_0^{\infty} e^{-st} f(t)\, dt$$

Here f(t) = $\boldsymbol{e^{at}}$

L{e^{at}}= $\int_0^{\infty} e^{-st} \boldsymbol{e^{at}}$ dt = $\int_0^{\infty} e^{-t(s-a)}$ dt = $\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^{\infty}$ = $\frac{-1}{s-a}(e^{-\infty} - e^{0})$ = $\frac{-1}{s-a}$ (o – 1)

L{$\boldsymbol{e^{at}}$} = $\frac{1}{s-a}$; a > 0

If a = 0, L{1} = $\frac{1}{s}$; (s > 0)

Similarly, L{$\boldsymbol{e^{-at}}$} = $\frac{1}{s+a}$

Similarly we can find Laplace Transformation of some standard functions listed bellow, these results can be easily proven using the standard definitions as mentioned in equation (I)

1) L{ $\sin at$} = $\frac{a}{s^2+a^2}$
2) L{$\cos at$} = $\frac{s}{s^2+a^2}$; s > 0
3) L{$\sinh at$} = $\frac{a}{s^2-a^2}$
4) L{$\cosh at$} = $\frac{s}{s^2-a^2}$; s > $|a|$
5) L{$\boldsymbol{t^n}$} = $\frac{\Gamma n+1}{s^{n+1}}$
6) L{t^n} = $\frac{n!}{s^{n+1}}$; If n is positive integer

Examples- 1) L{t^2} = $\frac{2!}{s^3}$; 2) L{$t^{3/2}$} = $\frac{\Gamma 3/2+1}{s^{3/2+1}} = \frac{\Gamma 5/2}{s^{5/2}}$; 3) L{ $\sin 3t$} = $\frac{3}{s^2+3^2}$; 4) L{$\cos 5t$} = $\frac{s}{s^2+5^2}$;

5) L{$\sinh 6t$} = $\frac{6}{s^2-6^2}$; 7) L{$\cosh 2t$} = $\frac{s}{s^2-2^2}$; 8) L{$\boldsymbol{e^{-t}}$} = $\frac{1}{s+1}$; 9) L{$\boldsymbol{e^{3t}}$} = $\frac{1}{s-3}$

Value of the Function $\overline{f}(s)$: By definition of Laplace Transform

1) $\int_0^{\infty} e^{-st} \sin t \, dt = \frac{1}{s^2+1^2} \Rightarrow \int_0^{\infty} e^{-7t} \sin t \, dt = \frac{1}{50}$; for s=7

2) $\int_0^{\infty} e^{-st} \cos 2t \, dt = \frac{s}{s^2+2^2} \Rightarrow \int_0^{\infty} e^{-3t} \cos 2t \, dt = \frac{3}{13}$; for s=2

Change of scale Property- If L{f(t)}= $\overline{f}(s)$ then, L{f(at)}= $\overline{f}(\frac{s}{a})$ is called frequency scaling property and L{f($\frac{t}{a}$)}= $\overline{f}(as)$ is called time scaling property.

Proof – By the definition, L{f(t)}= $\int_0^{\infty} e^{-st}$ f(t) dt

$$L\{f(at)\}= \int_0^{\infty} e^{-st} f(at)\, dt$$

Put at = u ⇒ a dt = du

When t = 0, u = 0 & when t = ∞, u = ∞

L{f(at)}= $\int_0^\infty e^{-sa/u}$ f(u) $\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-sa/u}$ f(u) du = $\frac{1}{a}\overline{f}(\frac{s}{a})$

For example, L{f(t)}= $\frac{3s}{s^2+4}$ then, L{f(4t)}= $\frac{1}{4}\overline{f}(\frac{s}{4}) = \frac{1}{4}\frac{3(\frac{s}{4})}{(s/4)^2+4} = \frac{3}{4}\frac{s}{s^2+64}$

First Shifting Theorem: - If L{f(t)}= $\overline{f}(s)$ then, L{e^{-at}f(t)}= $\overline{f}(s+a)$

Proof- By the definition, L{f(t)}= $\int_0^\infty e^{-st}$ f(t) dt

L{$\boldsymbol{e^{-at}}$f(t)}= $\int_0^\infty e^{-st}\,\boldsymbol{e^{-at}}$f(t) dt = $\int_0^\infty e^{-(s+a)t}$ f(t) dt = $\overline{f}(s+a)$

Similarly, L{$\boldsymbol{e^{at}}$f(t)}= $\overline{f}(s-a)$

For example, L{ $\sin at$} = $\frac{a}{s^2+a^2}$ ⇒ L{$\boldsymbol{e^{-bt}}$ sin at}= $\frac{a}{(s+b)^2+a^2}$

Second Shifting Theorem: - If L{f(t)}= $\overline{f}(s)$ and g(t) = $\begin{cases} f(t-a)\,; t > a \\ 0 \qquad ; t < a \end{cases}$

then L{g(t)} = $e^{-as}\,\overline{f}(s)$

Proof- By the definition, L{g(t)} = $\int_0^\infty e^{-st}$ g(t) dt

L{g(t)} = $\int_0^\infty e^{-st}$ g(t) dt = $\int_0^a e^{-st}$ g(t) dt + $\int_a^\infty e^{-st}$ g(t) dt = 0 + $\int_a^\infty e^{-st}$ f(t-a) dt

Put t-a = u ⇒dt = du

When t = a, u = 0 & when t = ∞, u = ∞

L{g(t)}= $\int_0^\infty e^{-s(a+u)}$ f(u) du = $e^{-as}\int_0^\infty e^{-su}$ f(u) du = $e^{-as}\,\overline{f}(s)$

For example, If f(t) = $\begin{cases} \cos\left(t-\frac{\pi}{3}\right); t > \frac{\pi}{3} \\ 0 \qquad\qquad ; t < \frac{\pi}{3} \end{cases}$ then find L{f(t)}.

From above theorem, L{f(t)} = $e^{-\frac{\pi}{3}}$ L{cos t} = $e^{-\frac{\pi}{3}}\left(\frac{s}{s^2+1}\right)$

Effect of Multiplication by t: - L{f(t)}= $\overline{f}(s)$, if we multiply powers of t with the original function f (t), the Laplace transform can be expressed as, L{t^nf(t)}= $(-1)^n\frac{d^n}{ds^n}$ [$\overline{f}(s)$].

Proof- This result can be proved by the using Mathematical Induction.

Step-1: To prove that the result is true for n=1.

By the definition, L{f(t)}= $\overline{f}(s)$ = $\int_0^\infty e^{-st}$ f(t) dt

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$\frac{d}{ds}$[$\overline{f}(s)$] = $\int_0^\infty \frac{\partial}{\partial s}$ [e^{-st} f(t)] dt = - $\int_0^\infty e^{-st}\, t$ f(t) dt = - L{t f(t)}

Thus, L{t f(t)} = (-1) $\frac{d}{ds}[\overline{f}(s)]$, which proves the result for n=1

Step-2: Let's assumed that the result is true when n is any natural number 'k'.

L$\{t^k$f(t)}= $(-1)^k \frac{d^k}{ds^k}[\overline{f}(s)]$

Step-3: To prove that the result holds true when n=k+1. From step 2

$(-1)^k \frac{d^k}{ds^k}[\overline{f}(s)]$= L$\{t^k$f(t)} = $\int_0^\infty e^{-st}\, t^k$ f(t) dt

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}[\overline{f}(s)] = \int_0^\infty \frac{\partial}{\partial s} e^{-st}\, t^k$ f(t) dt = $-\int_0^\infty e^{-st}\, t\, t^k$ f(t) dt = $-\int_0^\infty e^{-st}\, t^{k+1}$ f(t) dt

$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}[\overline{f}(s)]$ = - L{ t^{k+1} f(t)} , which proves the result for n=k+1.

Thus, by the rule of Mathematical Induction, it can be said that the result is true for any value of n.

Example: Find Laplace Transform of t sin at

Solution: - L{t f(t)} = (-1) $\frac{d}{ds}[\overline{f}(s)]$

L{t sin at} = (-1) $\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] = \frac{2as}{(s^2+a^2)^2}$

Value of the function L{f(t)}:

1) $\int_0^\infty e^{-4t}$ t sin t dt = L{t sin t} = (-1) $\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] = \frac{2s}{(s^2+1)^2} = \frac{8}{289}$; putting s=4
2) $\int_0^\infty e^{-t}$ t^3 sin t dt = L{t^3 sin t} = $(-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s^2+1^2}\right] = \frac{2(18s^3-6s^2-12s)}{(s^2+1)^4}$ = 0 ; putting s=1

Effect of Division by t: - If L{f(t)}= $\overline{f}(s)$, then the Laplace Transform when the function is divided by the variable can be expressed as, L$\left\{\frac{f(t)}{t}\right\}$= $\int_s^\infty \overline{f}$(s) ds.

Proof- By the definition of Laplace Transform, $\overline{f}(s) = \int_0^\infty e^{-st}$ f(t) dt

Integrating both sides with respect to s between the limits s to ∞ and then changing the order of integration on the RHS

$\int_s^\infty \overline{f}(s) = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt = \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$ = **L**$\left\{\frac{f(t)}{t}\right\}$

Example: Find Laplace Transform of $\frac{1}{t}$ (1- cos t).

Solution: - L$\left\{\frac{f(t)}{t}\right\}$ = $\int_s^\infty \overline{f}$(s) ds

L{$\frac{1}{t}$ (1- cos t)} = $\int_s^\infty [L\{1\} - L\{\cos t\}]$ ds=$\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]$ds = $\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

Value of the function L$\left\{\frac{f(t)}{t}\right\}$**:**

1) $\int_0^\infty \frac{e^{-2t}-e^{-3t}}{t}$ dt=$\int_s^\infty L\{e^{-2t} - e^{-3t}\}$ ds=$\int_s^\infty \left[\frac{1}{s+2} - \frac{1}{s+3}\right]$ds=log$\left(\frac{s+3}{s+2}\right)$=log$\left(\frac{3}{2}\right)$; putting s=0
2) $\int_0^\infty \frac{\cos 4t - \cos 3t}{t}$ dt=$\int_s^\infty L\{\cos 4t - \cos 3t\} ds$=$\int_s^\infty \left(\frac{s}{s^2+4^2} - \frac{s}{s^2+3^2}\right)$ds= log$\left(\frac{3}{4}\right)$; putting s=0

Laplace Transform of Derivatives: Let f (t) be the function of t & f'(t) be continuous & L{f(t)}= $\overline{f}(s)$, then the Laplace Transform of its derivative can be expressed as, L{f'(t)}= - f(0) + s L{f(t)} ; provided $\lim_{t\to\infty}[e^{-st}f(t)]=0$

Proof- By the definition of Laplace Transform, L{f'(t)}= $\int_0^\infty e^{-st}\, f'(t)\, dt$ …(I)

Integrating by parts, L{f'(t)}=$[e^{-st}f(t)]_0^\infty$- $\int_0^\infty(-s)e^{-st}f(t)dt$= -f(0)+s $\int_0^\infty e^{-st}$ f(t) dt= -f(0)+sL{f(t)} Differentiating (I) again, we get ,

L{f''(t)}= -f'(0)+s[L{f'(t)}]= -f'(0)+s[f(0)+sL{f(t)}]= -f'(0)+sf(0)+s^2L{f(t)}

In general, L{f^n(t)}=$s^n\overline{f}(s)$ - s^{n-1}f(0) - $s^{n-2}f'(0)$ - $s^{n-3}f''(0)$ - $s^{n-4}f'''(0)$ - - - -$f^{(n-1)}(0)$; assuming that $\lim_{t\to\infty}[e^{-st}f^m(t)]=0$; m=0, 1, 2 ….(n-1)

These results will be used in the solution of differential equations.

Example- If f(t) = $\frac{\sin t}{t}$ then, find L{f'(t)}

Solution: - L$\left\{\frac{\sin t}{t}\right\}$ = $\int_s^\infty L\{\sin t\}\, ds$=$\int_s^\infty \frac{1}{s^2+1}$ds=$[\tan^{-1}s]_s^\infty$= $\frac{\pi}{2}-\tan^{-1}s = \cot^{-1}s$

Now, L{f'(t)}= sL{f(t)} - f(0) = s $\cot^{-1}s - 0$ = s $\cot^{-1}s$

Laplace Transform of an Integral: If L{f(t)}= $\overline{f}(s)$, when the function f(t) is integrated, its Laplace Transform can be expressed as, L$\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\overline{f}(s)$.

Proof- By definition of Laplace Transform

L$\left\{\int_0^t f(u)du\right\}$ = $\int_0^\infty e^{-st}\left[\int_0^t f(u)du\right]$ dt ; integrating by parts

L$\left\{\int_0^t f(u)du\right\}$ = $\left[\int_0^t f(u)du\left(-\frac{e^{-st}}{s}\right)\right]_0^\infty$ - $\int_0^\infty\left[\left(-\frac{e^{-st}}{s}\right)\frac{d}{dt}\int_0^t f(u)du\right]$dt ; but $\frac{d}{dt}\int_0^t f(u)du = f(u)$

L$\left\{\int_0^t f(u)du\right\}$ = $\int_0^\infty \frac{1}{s}\, e^{-st}f(t)dt = \frac{1}{s}\int_0^\infty e^{-st}f(t)dt$ = $\frac{1}{s}$L{f(t)} = $\frac{1}{s}\overline{f}(s)$.

The above result can be generalized as,

L$\left\{\int_0^t\int_0^t \dots\dots\dots\dots\int_0^t f(u)(du)^n\right\}$ = $\frac{1}{s^n}$ L{f(t)}

Example- Find Laplace Transform of $\int_0^t \sin 3u\, du$

Solution: - By definition, L$\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\overline{f}(s) = \frac{1}{s}\, L\{f(t)\}$

L$\left\{\int_0^t \sin 3u\, du\right\} = \frac{1}{s}\, L\{\sin 3t\}$= $\frac{1}{s}\left(\frac{3}{s^2+3^2}\right) = \frac{3}{s(s^2+9)}$

II. Inverse Laplace Transform

Definition- If L{f(t)} = $\overline{f}(s)$, then f(t) is called Inverse Laplace Transform of $\overline{f}(s)$ and is denoted by $L^{-1}\{\overline{f}(s)\}$=f(t). Here L^{-1} denotes the Inverse Laplace Transform.

We get the following Inverse Laplace Transform of some standard functions from definition (I)

i) $L\{1\} = \frac{1}{s}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s}\right\} = 1$

ii) $L\{e^{-at}\} = \frac{1}{s+a}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

iii) $L\{e^{at}\} = \frac{1}{s-a}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

iv) $L\{\sin at\} = \frac{a}{s^2+a^2}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

v) $L\{\cos at\} = \frac{s}{s^2+a^2}$; $s > 0$ $\Rightarrow$ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

vi) $L\{\sinh at\} = \frac{a}{s^2-a^2}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

vii) $L\{\cosh at\} = \frac{s}{s^2-a^2}$; $s > |a|$ $\Rightarrow$ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cosh at$

viii) $L\{t^{n-1}\} = \frac{\Gamma n}{s^n}$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

Methods of Obtaining Inverse Laplace Transforms: There are various methods of obtaining Inverse Laplace Transform. Some of them discussed bellow.

(a) Use of above standard results

Example- Find Inverse Laplace Transform of $\frac{3}{s} + \frac{1}{s^3} + \frac{1}{s-4}$

Solution:- $L^{-1}\left\{\frac{3}{s} + \frac{1}{s^3} + \frac{1}{s-4}\right\} = 3L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\} + L^{-1}\left\{\frac{1}{s-4}\right\} = 3 + \frac{t^2}{2} + e^{4t}$

(b) Using Shifting Theorem

If $L\{f(t)\} = \overline{f}(s)$, then $L\{e^{-at}f(t)\} = \overline{f}(s+a)$. This means if $f(t) = L^{-1}\{\overline{f}(s)\}$ then $L^{-1}\{\overline{f}(s+a)\} = e^{-at}f(t)$. i.e $L^{-1}\{\overline{f}(s+a)\} = e^{-at} L^{-1}\{\overline{f}(s)\}$

Example- i) $L^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t} L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-3t}\, t$ ii) $L^{-1}\left\{\frac{(s+2)}{(s+2)^2+9}\right\} = e^{-2t}\cos 3t$

(c) Method of Partial Fraction

Whenever possible, it is always easier to solve a problem on Inverse Laplace Transform by expressing the given function $\overline{f}(s)$ into a sum of linear or quadratic partial fraction as, $\overline{f}(s) = \frac{A}{(s+a)^r} + \frac{Bs+C}{(s^2+a^2)^r}$ and then use standard results to find L^{-1}.

Example- Find Inverse Laplace Transform of $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$

Solution:- Let $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$ … by partial fraction

$2s^2 - 4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$

Put s+1=0 $\Rightarrow$ s = -1 $\Rightarrow$ A= $\frac{-1}{6}$, put s-2=0 $\Rightarrow$ s=2 $\Rightarrow$ B = $\frac{-4}{3}$, put s-3=0 $\Rightarrow$ s=3 $\Rightarrow$ C = $\frac{7}{2}$

Thus $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-1}{6}\frac{1}{s+1} - \frac{4}{3}\frac{1}{s-2} + \frac{7}{2}\frac{1}{s-3}$

$L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \frac{-1}{6}L^{-1}\left(\frac{1}{s+1}\right) - \frac{4}{3}L^{-1}\left(\frac{1}{s-2}\right) + \frac{7}{2}L^{-1}\left(\frac{1}{s-3}\right) = \frac{-1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$

(d) By using Convolution theorem

Definition:-If f(t) and g(t) are causal functions then their convolution(twisting together) is defined by:

$(f * g)(t) = \int_0^t f(u)g(t-u)\,dt$

Theorem:- If $L\{f(t)\} = \overline{f}(s)$ and If $L\{g(t)\} = \overline{g}(s)$ then $L^{-1}\{\overline{f}(s).\overline{g}(s)\} = \int_0^t f(u).g(t-u)du$

Example- Find $L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$ by using convolution theorem.

Solution:- Let $\overline{f}(s) = \frac{1}{s+3}$ & $\overline{g}(s) = \frac{1}{s-1}$

$L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t} = f(t)$ & $L^{-1}\{\overline{g}(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t} = g(t)$

By convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = L^{-1}\left\{\frac{1}{s+3}\frac{1}{s-1}\right\} = \int_0^t f(u)g(t-u)dt \quad = \int_0^t e^{-3u}e^{t-u}du = e^t\int_0^t e^{-4u}du = e^t\left[\frac{e^{-4u}}{(-4)}\right]_0^t$$
$$= \frac{e^t}{-4}(e^{-4t}-1) = \frac{-1}{4}(e^{-3t}-e^t)$$

Laplace Transform of Periodic Function:

If f(t) is periodic function with period a i.e $L\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st}$ f(t) dt.

Example- Find Laplace Transform of f(t) = $\frac{Ct}{T}$ for $0< t < T$ & f(t) = f(t+T)

Solution:- Since f(t) is periodic function with period T

By definition, $L\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^T e^{-st} f(t)\, dt = \frac{1}{1-e^{-sa}}\int_0^T e^{-st}\frac{Ct}{T}\, dt = \frac{1}{1-e^{-sa}}\frac{C}{T}\int_0^T e^{-st}t\ dt$

$=\frac{1}{1-e^{-sa}}\frac{C}{T}\left[t\left(\frac{e^{-st}}{-s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^T = \frac{1}{1-e^{-sa}}\frac{C}{T}\left[\frac{1}{s^2}(1-e^{-sT})-\frac{Te^{-sT}}{s}\right] = C\left[\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-sT})}\right]$

Laplace Transform of Heaviside's Unit-Step Function:

Definition- The unit step function or Heaviside's unit-step function is defined as,

H(t-a) = $\begin{cases} 0 & for\ t < a \\ 1 & for\ t \geq a \end{cases}$ where a≥ 0, is called Heaviside's unit-step function.

In particular, if a = 0 ; H(t) = $\begin{cases} 0 & for\ t < 0 \\ 1 & for\ t \geq 0 \end{cases}$

Laplace Transform of H(t-a)-

By the definition of Laplace Transform, $L\{f(t)\} = \int_0^\infty e^{-st}$ f(t) dt

$L\{H(t-a)\} = \int_0^\infty e^{-st}$ H(t-a) dt = $\int_0^a e^{-st}(0) + \int_a^\infty e^{-st}(1) = \left[\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$

$L^{-1}\left[\frac{e^{-st}}{s}\right]$ = H(t-a) & if a = 0 L{H(t)} = $\frac{1}{s}$ and $L^{-1}\left[\frac{1}{s}\right]$ = H(t)

Example- Find Laplace Transform of $(t-1)^2$H(t-1).

Solution:- Comparing $(t-1)^2$H(t-1) with f(t-a)H(t-a) we get a = 1 & f(t) = t^2

$L\{f(t)\} = \overline{f}(s) = L\{t^2\} = \frac{2}{s^3}$

By second shifting property, $L\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$

$L\{(t-1)^2H(t-1)\} = e^{-s}\frac{2}{s^3} = \frac{2e^{-s}}{s^3}$

Dirac-delta Function or Unit Impulse Function-

Definition-The function represented by the figure can be defined as,

$$F(t) = \begin{cases} 0 & t < a \\ \frac{1}{\epsilon} & 0 \le t \le a+\epsilon, \\ 0 & t > a+\epsilon \end{cases}$$

As $\epsilon \to \infty$, the function F(t) tends to infinity at a, and is zero everywhere else. But the integral of F(t) is unity. Thus, the $\lim_{t\to\infty} \int_0^{\infty} F(t)dt \ (= 1)$ represents a unit impulse at t = a. Hence the limiting form of F(t) as $\epsilon \to 0$, is called as the Unit Impulse Function or the Dirac Delta Function and is denoted by $\delta(t-a)$.

$$\delta(t-a) = \lim_{t\to\infty} F(t)$$

Laplace Transform of Dirac-delta Function-

$L[\delta(t-a)] = e^{-as} \Rightarrow L[\delta(t)] = 1$; if a= 0

Also, $L^{-1}[e^{-as}] = \delta(t-a)$ & $L^{-1}[1] = \delta(t)$

Note: $L[f(t)\delta(t-a)] = e^{-as}f(a)$

Example- Find Laplace Transform of $\sin 2t \ \delta(t-2)$

Solution:- Here f(t) = sin 2t and a = 2

$L[\sin 2t \ \delta(t-2)] = e^{-2s}f(2) = e^{-2s}\sin 4$

Applications of Laplace Transform

Laplace Transform can be used to solve ordinary differential equations. The Laplace Transform reduces linear differential equations to an algebraic equation, which can then solved by the formal rules of algebra. The original equation can then solved by applying inverse Laplace Transform.

Example- Solve $(D^2 - D - 2)y = 20sin2t$, with y (0) =1 and $y'(0) = 2$.

Solution:- Let L(y) = $\bar{y}$. Taking Laplace Transform on both sides of given differential equation

$L(y'') - L(y')$-2L(y)=20 L(sin 2t); but $L(y')=s\bar{y}-y(0)= s\bar{y} - 1$ and

$L(y'')= s^2\bar{y}-sy(0)- y'(0) = s^2\bar{y}-s-2$. The equation becomes

$$(s^2\bar{y}-s-2) - (s\bar{y} - 1) - 2\bar{y} = 20\frac{2}{s^2+4}$$

$$(s^2-s-2)\ \bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3+s^2+4s+44}{s^2+4}$$

$$\bar{y} = \frac{s^3+s^2+4s+44}{(s^2+4)((s^2-s-2)} = \frac{-8}{3}\frac{1}{s+1} + \frac{8}{3}\frac{1}{s-2} + \frac{s-6}{s^2+4}$$ (by partial fraction).Taking inverse laplace transform

$y = \frac{-8}{3}e^{-t} + \frac{8}{3}e^{-2t} + \cos 2t - 3\sin 2t$, is the required solution.

Conclusion-

Laplace Transform is powerful tool in different areas of mathematics and engineering. This paper gives a very basic information and idea of Laplace Transform. Engineering students will satisfy with this basic idea of Laplace Transform. The motive of this paper is to present the clear Study regarding Laplace Transformation with their important application. The Study on Properties and Applications of this Laplace Transformation method shows how it could be useful for finding the solutions for different problems of engineering.

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