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BANACH SPACE VALUED SEQUENCE SPACE INVOLVING LACUNARY SEQUENCE

Indu Bala
Department of Mathematics,
Government College,
Chhachhrauli (Yamuna Nagar) 135 103, INDIA

Abstract. The main object of the paper is to introduce a new Banach space valued sequence space $ces_\theta(X,p)$. Various algebraic and topological properties of the space have been examined. Some inclusion relations between the space have been investigated. Our results generalize and unify the corresponding earlier results of Karakaya[2], Shiue[7], Sanhan and Suantai[6].
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1. Introduction

By a lacunary sequence $\theta = (k_r); r = 0,1,2,...,$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be denoted by q_r . The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [1] as follows:

$N_\theta = \{x = (x_k): \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - l| = 0 \text{ for some } l\}.$

There is a strong connection [9] between N_θ and the space w of strongly Cesàro summable sequences, which is defined by

$w = \{x = (x_k): \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |x_k - l| = 0 \text{ for some } l\}.$

In the special case where $\theta = (2^r)$, we have $N_\theta = w$. Infact, for a lacunary sequence θ, $N_\theta = w$ if and only if $1 < \liminf_r q_r \le \limsup_r q_r < \infty$[1, p. 511].

Let w, ℓ^0 denote the spaces of all scalar and real sequences, respectively. For $1 < p < \infty$, the Cesàro sequence space ces_p defined by

$$ces_p = \{x \in \ell^0: \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|\right)^p < \infty\}$$

is a Banach space when equipped with the norm

$$\| x \| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}$$

This space was first introduced by Shiue [7], which is useful in the theory of Matrix operator and others (see [3, 4]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors.

Sanhan and Suantai [6] introduced and studied a generalized Cesàro sequence space $ces(p)$, where $p = (p_n)$ is a bounded sequence of positive real numbers.

Quite recently, Karakaya[2], Ozturk and Basarir[5] introduced a new sequence space involving lacunary sequence and examined some geometric properties of this space equipped with Luxemburg norm.

Let $(X, \|.\|)$ be a Banach space over the complex field $\mathbb{C}$. Denote by $w(X)$ the space of all X-valued sequences. Let $p = (p_k)$ be a bounded sequence of positive real numbers. We now introduce the Banach space valued sequence space $ces_\theta(X, p)$ as follows.

$$ces_\theta(X, p) = \left\{ x \in w(X): \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} \|\mathrm{x_k}\| \right)^{p_r} < \infty \right\}.$$

Some well-known spaces are obtained by specializing X and p.

(i) If $X = \mathbb{C}$, then $ces_\theta(X, p) = \ell(p, \theta)$ (Karakaya [2]).

(ii) If $X = \mathbb{C}$, $p_n = p (1 \leq p < \infty)$ for all n and $\theta = (2^r)$, then $ces_\theta(X, p) = ces_p$ (Shiue [7]).

(iii) If $X = \mathbb{C}$ and $\theta = (2^r)$, then $ces(f, p) = ces(p)$ (Sanhan and Suantai [6]).

The following inequalities are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k$, then for any complex a_k and b_k,

(1) $|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$,

where $C = \max(1, 2^{H-1})$. Also for any complex λ,

(2) $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$.

2. Linear topological structure of $ces_\theta(X, p)$ space

In this section we establish some algebraic and topological properties of the sequence space defined above. In order to discuss the properties of $ces_\theta(X, p)$, we assume that (p_n) is bounded.

Theorem 2.1. $ces_\theta(X, p)$ is a linear space over the complex field $\mathbb{C}$.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. $ces_\theta(X,p)$ is a topological linear space, paranormed by

$$g(x)=\left(\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{p_r}\right)^{\frac{1}{M}} \qquad (2.1)$$

where $H=\sup p_r<\infty$ and $M=\max(1,H)$.

The proof follows by using standard techniques and the fact that every paranormed space is a topological linear space [8, p. 37].

Corollary 2.3. If p is a constant sequence, then $ces_\theta(X,p)$ is a normed space for $p\geq 1$ and a p-normed space for $p<1$.

Theorem 2.4. $ces_\theta(X,p)$ is a Fréchet space paranormed by (2.1).

Proof. In view of Theorem 2.2 it suffices to prove the completeness of $ces_\theta(X,p)$. Let $\left(x^{(i)}\right)$ be a Cauchy sequence in $ces_\theta(X,p)$. Then $g\left(x^{(i)}-x^{(j)}\right)\to 0$ as $i,j\to\infty$, that is

$$\left(\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}||x_k^{(i)}-x_k^{(j)}||\right)^{p_r}\right)^{\frac{1}{M}}\to 0,\ \text{as}\quad i,j\to\infty, \qquad (2.2)$$

which implies that for each fixed k, $\left\|x_k^{(i)}-x_k^{(j)}\right\|\to 0$ as $i,j\to\infty$ and so $\left(x_k^{(i)}\right)$ is a Cauchy sequence in X for each fixed k. Since X is complete, there exists a sequence $x=(x_k)$ such that $x_k\in X$ for each $k\in\mathbb{N}$ and $\lim_{i\to\infty}x_k^i=x_k$ for each k. Now from (2.2), we have for $\epsilon>0$, there exists a natural number K such that

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}-x_k^{(j)}\right\|\right)^{p_r}<\epsilon^M \quad \text{for}\quad i,j>K. \qquad (2.3)$$

Since for any fixed natural number r_0, we have from (2.3),

$$\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}-x_k^{(j)}\right\|\right)^{p_r}<\epsilon^M \quad \text{for}\quad i,j>K,$$

by taking $j\to\infty$ in the above expression we obtain

$$\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}-x_k\right\|\right)^{p_r}<\epsilon^M \quad \text{for}\quad i>K.$$

Since r_0 is arbitrary, by taking $r_0\to\infty$, we obtain

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}-x_k\right\|\right)^{p_r}<\epsilon^M \quad \text{for}\quad i>K,$$

that is, $g\left(x^{(i)}-x\right)<\epsilon$ for $i>K$.

To show that $x \in ces_\theta(X,p)$, let $i > K$ and fix r_0. Since $p_r/M \leq 1$ and $M \geq 1$, using Minkowski's inequality, we have

$$\left(\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{p_r}\right)^{\frac{1}{M}} = \left(\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k - x_k^{(i)} + x_k^{(i)}\right\|\right)^{p_r}\right)^{\frac{1}{M}}$$

$$\leq \left(\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k - x_k^{(i)}\right\|\right)^{p_r}\right)^{\frac{1}{M}} + \left(\sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}\right\|\right)^{p_r}\right)^{\frac{1}{M}}$$

$$< \epsilon + g\left(x^{(i)}\right),$$

from which it follows that $x \in ces_\theta(X,p)$ and the space is complete.

Corollary 2.5. If p is a constant sequence and $p \geq 1$, then $ces_\theta(X,p)$ is a Banach space.

3. Inclusion between $ces_\theta(X,p)$ spaces

We now investigate some inclusion relations between $ces_\theta(X,p)$ spaces.

Theorem 3.1. If $p = (p_r)$ and $q = (q_r)$ are bounded sequences of positive real numbers with $0 < p_r \leq q_r < \infty$ for each r, then $ces_\theta(X,p) \subseteq ces_\theta(X,q)$.

Proof. Let $x \in ces_\theta(X,p)$. Then $\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{p_r} < \infty$. This implies that

$\frac{1}{h_r}\sum_{k\in I_r}\|x_k\| \leq 1$ for sufficiently large values of r, say $r \geq r_0$ for some fixed $r_0 \in \mathbb{N}$. Since $p_r \leq q_r$, we have

$$\sum_{r\geq r_0}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{q_r} \leq \sum_{r\geq r_0}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{p_r} < \infty.$$

This shows that $x \in ces_\theta(X,q)$ and completes the proof.

Theorem 3.2. If (t_r) and (u_r) are bounded sequences of positive real numbers with

$0 < t_r, u_r < \infty$ and if $p_r = \min(t_r,u_r)$, $q_r = \max(t_r,u_r)$, then c

$es_\theta(X,p) = ces_\theta(X,t) \cap ces_\theta(X,u)$ and $ces_\theta(X,q) = G$ where G is the subspace of $w(X)$ generated by $ces_\theta(X,t) \cup ces_\theta(X,u)$.

Proof. It follows from Theorem 3.1 that $ces_\theta(X,p) \subseteq ces_\theta(X,t) \cap ces_\theta(X,u)$ and that

$G \subseteq ces_\theta(X,q)$. For any complex λ, $|\lambda|^{p_r} \leq \max(|\lambda|^{t_r}, |\lambda|^{u_r})$; thus $ces_\theta(X,t) \cap ces_\theta(X,u) \subseteq ces_\theta(X,p)$. Let $A = \{r: t_r \geq u_r\}$ and $B = \{r: t_r < u_r\}$. If $x \in ces_\theta(X,q)$, we write

$$y_r = x_r (r \in A) \quad \text{and} \quad y_r = 0 (r \in B); and$$
$$z_r = 0 (r \in A) \quad \text{and} \quad z_r = x_r (r \in B).$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|y_k\|\right)^{t_r} = \sum_{r\in A} + \sum_{r\in B} = \Sigma\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)^{q_r} < \infty,$$

and so $y \in ces_\theta(X,t) \subseteq G$. Similarly, $z \in ces_\theta(X,t) \subseteq G$. Thus, $x = y + z \in G$. We have proved that $ces_\theta(X,q) \subseteq G$, which completes the proof.

Corollary 3.3. The three conditions $ces_\theta(X,t) \subseteq ces_\theta(X,u)$, $ces_\theta(X,p) = ces_\theta(X,t)$ and $ces_\theta(X,q) = ces_\theta(X,u)$ are equivalent.

Corollary 3.4. $ces_\theta(X,t) = ces_\theta(X,u)$ if and only if $ces_\theta(X,p) = ces_\theta(X,q)$.

REFERENCES

1. A. R. Freedman, J. J. Sember and M. Raphael, Some Cesàro type summability spaces, Proc. London Math. Soc., 37(3) (1978), 508-520.

2. V. Karakaya, Some geometric properties of sequence spaces involving lacunary sequence, J. Inequal. Appl., (2007), Article ID 81028, 8 pages, doi:10.1155/2007/81028.

3. P. Y. Lee, Cesàro sequence spaces, Math. Chronicle, 13(1984), 29-45.

4. Y. Q. Lui, B. E. Wu and P. Y. Lee, Method of Sequence Spaces, Guangdong of Science and Technology Press, 1996(Chinese).

5. M. Ozturk and M. Basarir, On $k - NUC$ property in some sequence spaes involving lacunary sequence, Thai J. Math., 5(1)(2007), 127-136.

6. W. Sanhan and S. Suantai, On k-nearly uniform convex property in generalized Cesàro sequence spaces, International J. Math. Math. Sci., 57(2003), 3599-3607.

7. J. S. Shiue, On the Cesàro sequence spaces,Tamkang J. Math., 1(1970), 19-25.

8. A. Wilansky, Modern Methods in Topological Vector Spaces, McGraw-Hill, 1978.