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A Mathematical Modelling of two phase coronary blood flow in arteries during Silent Ischemia

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Abstract

In this paper, we will concentrate on the nature of coronary blood flow in arteries during Silent ischemia. Blood is considered as Non-Newtonian two-phase mixture of plasma and red blood cells. We have applied the Power law Non-Newtonian model, represented by the equation of continuity and the equation of motion. We have collected pathological data of Silent ischemia patients for the graphical study of blood pressure drop versus hematocrit. Including everything the presentation is in tensorial form and solution techniques adopted is analytical as well as numerical.

Keywords: Two phase blood flow, coronary blood flow, Non-Newtonian Power law model, Hematocrit, Blood pressure drop.

1. Introduction

The Coronary arteries supply blood to the heart muscle. Like all other tissues of the body, the heart muscle requires oxygen-rich blood to function. Also, deoxygenated blood must be carried away. The coronary arteries wrap around the outside of the heart. Small branches dive into the heart muscle to bring it blood [16].There are two primary coronary arteries, the right coronary artery (RCA) and the left main coronary artery (LMCA). Both of these originate from the root of the aorta. The RCA originates from the anterior ascending aorta and supplies blood primarily to the right atrium, right ventricle. The senatorial nodal artery is a branch of the RCA that supplies the SA node. The RCA also supplies the AV node through a septal performing branch in 90% of population.

Since coronary arteries send blood to the heart muscle, any coronary artery problem can cause serious health problems. It reduces the flow of oxygen and nutrients to the heart muscle. This can load to a heart attack and possibly death. The most common cause to heart disease is atherosclerosis. This is a buildup of plaque in the inner lining of an artery. It causes to artery to become narrow are blocked. The causes less blood to get to the heart tissues. [17]

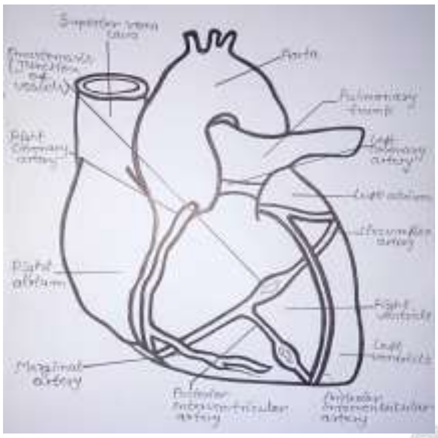


Fig 1: Structure and Function of Coronary arteries

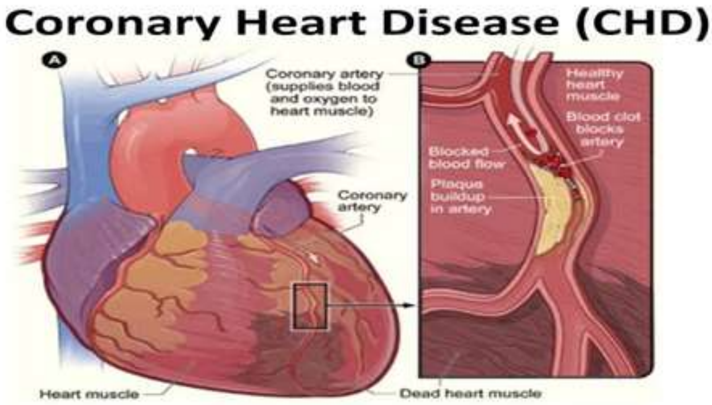


Fig: 2: Description of Silent Ischemia:

Ischemia is a condition in which the blood flow (and thus oxygen) is restricted or reduced in a part of the body. Ischemic heart disease is the term given to heart problems caused by narrowed heart arteries. When arteries are narrowed, less blood and oxygen reaches the heart muscle. This is also called coronary artery disease and coronary heart disease. This can ultimately lead to heart attack [6]. Silent ischemia occurs when the heart temporarily doesn't receive enough blood (and thus oxygen), but the person with the oxygen-deprivation doesn't notice any effects. Most silent ischemia occurs when one or more coronary arteries are narrowed by plaque. It can also occur when the heart is forced to work harder than normal. People who have diabetes or who have had a heart attack are most likely to develop Silent ischemia [9].

1.1.Constitution of Blood:

Blood is a body fluid in humans and other animals that delivers necessary substances such as nutrients and oxygen to the cells and transports metabolic waste products away from those same cells**.** Blood is a circulating tissue composed of fluid plasma and cells (red blood cells, white blood cells, platelets). Blood is composed of two parts-

1) Plasma, which constitutes 55% of total blood volume. Composed of 90% water, salts, lipids and hormones, it is especially rich in proteins, immunoglobulin, clotting factors and fibrinogen.
2) Formed cellular elements (RBC, WBC & Platelets) which combine to make the remaining 45% of blood volume. Whole blood (plasma and cells) exhibits Non-Newtonian fluid dynamics [18, 15].

The percentage of volume covered by blood cells in the whole blood is called hematocrit. The total volume concentration of leukocytes and thrombocytes is only about 1% which is negligible. Then we have considered only two phases of blood, which one phase is red blood cells and other phase is plasma. A hematocrit ranging from 42% to 52% in males and 35% to 47% in females is typically considered normal.

2. Real Model

2.1. Choice of frame of reference:

The frame of reference for mathematical model of the moving blood keeping in view the difficulty and generality of the problem of blood flow. We select three dimensional orthogonal curvilinear co-ordinate system, assigned as E^3 called as 3-dimensional Euclidean space. We interpret the quantities related to blood flow in tensorial form which is more realistic. Let the co-ordinate axis be OX^i where O is origin and i = 1, 2, 3. The mathematical description of the state is a moving blood is affected by means of functions which give the distribution blood velocity $v^k = v^k(x^i, t)$[11].

2.2. Choice of parameters:

Blood is Non-Newtonian fluids. If the relation between stress and strain rate are linear the flow is Newtonian otherwise Non-Newtonian.

The constitutive equations for fluids-

$$\tau = \eta e^n$$

If n=1 then the nature of fluid is Newtonian and if $n \neq 1$ then the nature of fluid is Non-Newtonian. Where τ is denoted by stress, e is denoted by strain rate, η is denoted by viscosity and n is the parameter (other than dependent and independent variable) depends upon nature of fluid. In present study there are five parameter are used but three components of velocity v^k , blood pressure P and density ρ.

2.3. Boundary conditions are as follows:

1. The velocity of blood flow on the axis of arteries at $r = 0$ will be maximum and finite, say v_0 = maximum velocity, $v = v_0$ then $A = 0$.
2. The velocity of blood flow on the wall of coronary artery at $r = R$, where R is the radius of coronary artery, will be zero. This condition is well known as no-slip condition. $v = 0 \;\; at \;\; r = R.$

2.4. Description of two phase blood volume:

Let the volume portion covered by blood cells in unit volume be x, x is replaced by $\frac{H}{100}$, where H is the hematocrit. Then the volume portion covered by the plasma will be (1-X).

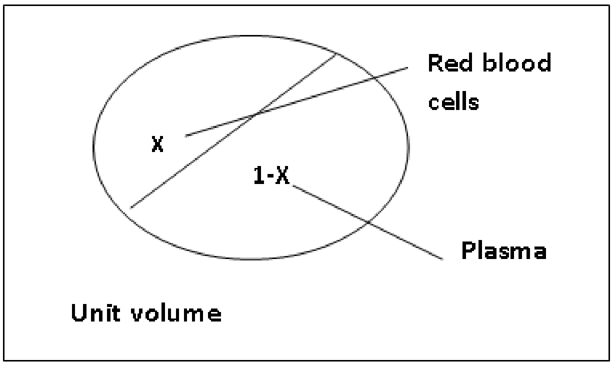


Fig 3: Unit Volume

3. Mathematical Formulation:

Blood is not an ideal fluid, it is in liquid form. Whenever the hematocrit increases, the effective viscosity of blood flowing in the arteries proximate to the heart depend upon the strain rate. In this situation, the blood flow becomes Non-Newtonian. In this model the concentration of red blood cells become high. In this situation the constitutive equation of blood is as follows:

$$T^{ij} = -pg^{ij} + \eta_m(e^{ij})^n = -pg^{ij} + T'^{ij} \quad \text{.........................} (3.1)$$

3.1. Law of Conservation of mass:

Blood flow must obey the principle of conservation of mass because there is no source and sink. Total flow is constant across all parts of circulatory system. On the other hand we say that "blood inflows on the circulatory system is equal to the blood outflows from the circulatory system.

Equation of continuity:

Equation of continuity for power law flow will be as follows

$$\frac{1}{\sqrt{g}(\sqrt{g}v^i)_{,i}} = 0 \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (3.2)$$

Equation of motion:

The equation of motion is extended as follows:

$$\rho_m \frac{\partial v^i}{\partial t} + \rho_m v^j v,^i_j = T^{ij}_{,j} \qquad \dots\dots\dots\dots\dots\dots\dots (3.3)$$

Where T^{ij} is taken from constitutive equation of power law flow (3.1). $\rho_m = X\rho_c + (1-X)\rho_P$ density of blood and $\eta_m = X\eta_C + (1-X)\eta_P$ is the viscosity of mixture of blood.$X = H/100$ is volume ratio of blood cells. H is hematocrit. Other symbols have their usual meanings.

Since the blood vessels are cylindrical, the above governing equations have to be transformed into cylindrical co-ordinates. As we know earlier:

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = z,$$

Matrix of metric tensor in cylindrical co-ordinates is as follows:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

While matrix of conjugate metric tensor is as follows:

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Whereas the Christoffel symbols of 2^{nd} kind are as follows:

$$\begin{Bmatrix} 1 \\ 2 \;\; 2 \end{Bmatrix} = -r, \qquad \begin{Bmatrix} 2 \\ 2 \;\; 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \;\; 2 \end{Bmatrix} = \frac{1}{r},$$

Relation between contravariant and physical components of velocity of blood flow will be as follows:

$$\sqrt{g_{11}}v^1 = v_r \;\Rightarrow\; v_r = v^1$$

$$\sqrt{g_{22}}v^2 = v_\theta \Rightarrow v_\theta = rv^2$$

and

$$\sqrt{g_{33}}v^3 = v_z \Rightarrow v_z = v^3$$

Again the physical components of $-p_{,j}g^{ij}$ are $-\sqrt{g_{ii}}p_{,j}g^{ij}$

The matrix of physical components of shearing stress-tensor $T'^{ij} = \eta_m(e^{ij})^n = \eta_m(g^{ik}v^i_{,k} + g^{ik}v^j_{,k.})^n$will be as follows:

$$\begin{bmatrix} 0 & 0 & \eta_m(dv/dr)^n \\ 0 & 0 & 0 \\ \eta_m(dv/dr)^n & 0 & 0 \end{bmatrix}$$

The covariant derivative of T'^{ij} is

$$T'^{ij}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}T^{ij}\right) + \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} T'^{kj}$$

Keeping in view the above facts, the governing tensorial equations can be transformed into cylindrical form which are as follows:

The equation of continuity -

$$\frac{\partial v}{\partial z} = 0 \qquad \dots\dots\dots\dots\dots (3.4)$$

The equation of motion –

r – component: $-\frac{\partial P}{\partial r} = 0$ …………… (3.5)

θ- component: $0 = 0$ …………...(3.6)

Z - component: $0 = -\frac{\partial p}{\partial z} + \frac{\eta_m}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial v_z}{\partial r}\right)^n\right)$ …………… (3.7)

Here this fact has been taken in view that the blood flow is axially symmetric in arteries concerned, i.e. $v_0 = 0$, and v_r, $v_z$and p do not depend upon θ.Also the blood flows steadily, i.e.

$$\frac{\partial p}{\partial t} = \frac{\partial v_r}{\partial t} = \frac{\partial v_\theta}{\partial t} = \frac{\partial v_z}{\partial t} = 0.$$

3.2. Solution: on integration equation (3.4) we get

$v_z = v(r)$ because v does not depend upon θ. …………….. (3.8)

The integration of equation of motion (3.5) yields:

$P = p(z)$, since p does not depend upon θ. …………….. (3.9)

Now, with the help of equations (3.8) and (3.9), the equation of motion (3.7) converts in the following form:

$$0 = -\frac{dp}{dz} + \frac{\eta_m}{r}\frac{d}{dr}\left(r\left(\frac{dv}{dr}\right)^n\right) \quad \text{…………….. (3.10)}$$

The pressure gradient $-(dp/dz) = P$ of blood flow in the arteries remote heart can be supposed to be constant and hence the equation (3.10) takes the following form:

$$\frac{d}{dr}\left(r\left(\frac{dv}{dr}\right)^n\right) = -\frac{Pr}{\eta_m} \quad \text{……………… (3.11)}$$

On integration the equation (3.11), we get

$$r\left(\frac{dv}{dr}\right)^n = -\frac{Pr^2}{2\eta_m} + A \quad \text{……………... (3.12)}$$

We know that the velocity of blood flow on the axis of cylindrical arteries is maximum and constant. So that we apply the boundary condition: at $r = 0, v = V_0$(constant), on equation (3.12) to get the arbitrary constant $A = 0$. Hence the equation (3.12) takes the following form:

$$r\left(\frac{dv}{dr}\right)^n = -\frac{Pr^2}{2\eta_m} \Rightarrow -\frac{dv}{dr} = \left(\frac{Pr}{2\eta_m}\right)^{1/n} \quad \text{……………… (3.13)}$$

Integrating equation (3.13) once again, we get

$$v = -\left(\frac{P}{2\eta_m}\right)^{1/n}\frac{r^{1/n+1}}{(n+1)/n} + B \quad \text{……………… (3.14)}$$

To determine the arbitrary constant B, we apply the no-slip condition on the inner wall of the arteries: at $r = 0,\ v = 0$, where R = radius of vessel, on equation (3.14) so as to get

$$B = \frac{P}{2\eta_m}^{1/n}\frac{nR^{1/n+1}}{(n+1)}$$

Hence the equation (3.14) takes the following form:

$$v = \frac{P}{2\eta_m}^{1/n}\frac{n}{(n+1)}\left(R^{1/n+1} - r^{1/n+1}\right) \quad \text{…………….. (3.15)}$$

Which determines the velocity of blood flow in the arteries remote from the heart where P is gradient of blood pressure and η_m is the viscosity of blood mixture [13].

4.Bio-physical Interpretation/ Result and Discussion:

The total flow-flux of blood through the transvers section of the arteries is:

$$Q = \int_0^R v.2\pi r = \int_0^R \left(\frac{P}{2\eta_m}\right)^{1/n}\frac{n}{n+1}\left(R^{1/n+1} - r^{1/n+1}\right)2\pi r\,dr$$

$$Q = \left(\frac{P}{2\eta_m}\right)^{1/n} \frac{n.2\pi}{n+1}\left(\frac{R^{1/n+1}.r^2}{2} - \frac{n.r^{1/n+3}}{3n+1}\right)_0^R$$

$$= \left(\frac{P}{2\eta_m}\right)^{1/n} \frac{n.2\pi}{n+1}\frac{(n+1)R^{1/n+3}}{2(3n+1)}$$

$$Q = \left(\frac{P}{2\eta_m}\right)^{1/n} \frac{\pi n\, R^{1/n+3}}{(3n+1)} \qquad \text{.................... (3.16)}$$

The pattern of blood flow can be shown by figure

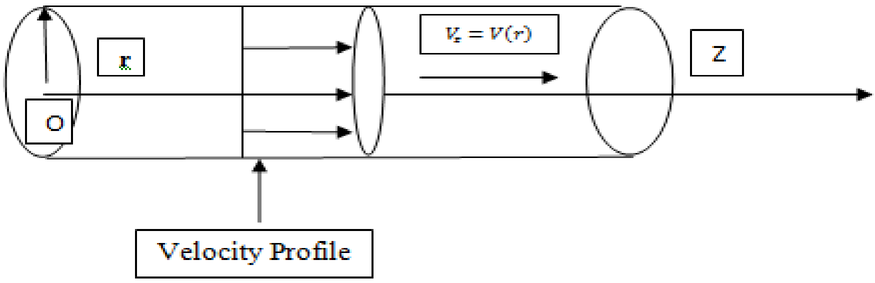


Fig 4: Pattern of blood flow

4.1. Patient case history-

Patient name: Raghav Ram Yadav **Age:** 58 **Sex:** Male

Diagnosis: Silent Ischemia (Coronary disease)

Date	Hemoglobin (HB)	Hematocrit (3 HB)	Blood Pressure (mmhg)	Arteries Pressure Drop in pascal second $\Delta P = S - \frac{S+D}{2}$
30/09/2021	12.4	0.035095	130/80	3333.05
01/10/2021	13.4	0.037925	120/80	2666.44
02/10/2021	13.7	0.038774	130/70	3999.66
05/10/2021	14.5	0.041038	120/80	2666.44

Table 4.1

Hematocrit (H) = 0.035095, Viscosity of mixture $\eta_m = 0.035\ pascal\ sec$[4]., Viscosity of plasma $\eta_p = 0.0015\ pascal\ sec$[12].

We know that $\eta_m = \eta_c X + \eta_p(1 - X)$, where $X = H/100 = 0.00035095$

$$0.035 = \eta_c(0.00035095) + 0.0015(0.99964905)$$

$$0.035 = 0.0372\,\eta_c + 0.001499474$$

$$\eta_c = 0.033500526\ pascal\ sec.$$

Again using this relation and change into the hematocrit

$$\eta_m = \eta_c X + \eta_p(1 - X)$$

$$= 0.95456693H + 0.001499474$$

From equation (3.16), Put $P = -\frac{dp}{dz}$, we get

$$Q = \left[\frac{\Delta P}{2\eta_m \Delta Z}\right]^{1/n} \frac{n\pi R^{1/n+3}}{3n+1} \qquad \text{...................... (3.17)}$$

$Q = 250\frac{ml}{m} = 0.004166\ \frac{m^3}{s}$ [2]

Length of coronary artery $\Delta Z = 50\ cm = 0.5m$ [3]

Radius of coronary artery R = 0.2 cm = 0.002 m [3]

Put the value of $Q,\ \Delta P,\ \Delta Z\ and\ R$ in equation (3.17)

$$0.004166 = \left[\frac{3333.05}{2 \times 0.035 \times 0.5}\right]^{1/n} \frac{n \times 3.14 \times (0.002)^{1/n+3}}{3n+1}$$

$$0.004166 = [95230]^{1/n} \left(\frac{n}{3n+1}\right) \times 3.14 \times (0.002)^3 \times (0.002)^{1/n}$$

$$0.004166 = (95230)^{1/n} \frac{n}{3n+1} \times 3.14 \times 0.000000008$$

$$165843.949045 = (95230)^{1/n} \frac{n}{3n+1}$$

By using trial & error method, we get the value of n is

$$\boldsymbol{n = 0.382}$$

Again using from equation (3.17)

$$Q = \left[\frac{\Delta P}{2\eta_m \Delta Z}\right]^{1/n} \frac{n\pi R^{\frac{1}{n}+3}}{3n+1}$$

$$0.004166 = \left(\frac{\Delta P}{\eta_m}\right)^{1/n} \left(\frac{1}{2 \times 0.5}\right)^{1/n} \left[\frac{0.382 \times 3.14 \times (0.002)^{\frac{1}{0.382}+3}}{3 \times 0.382 + 1}\right]$$

$$\Delta P = (0.95456693H + 0.001499474) \times (1.99847544)$$

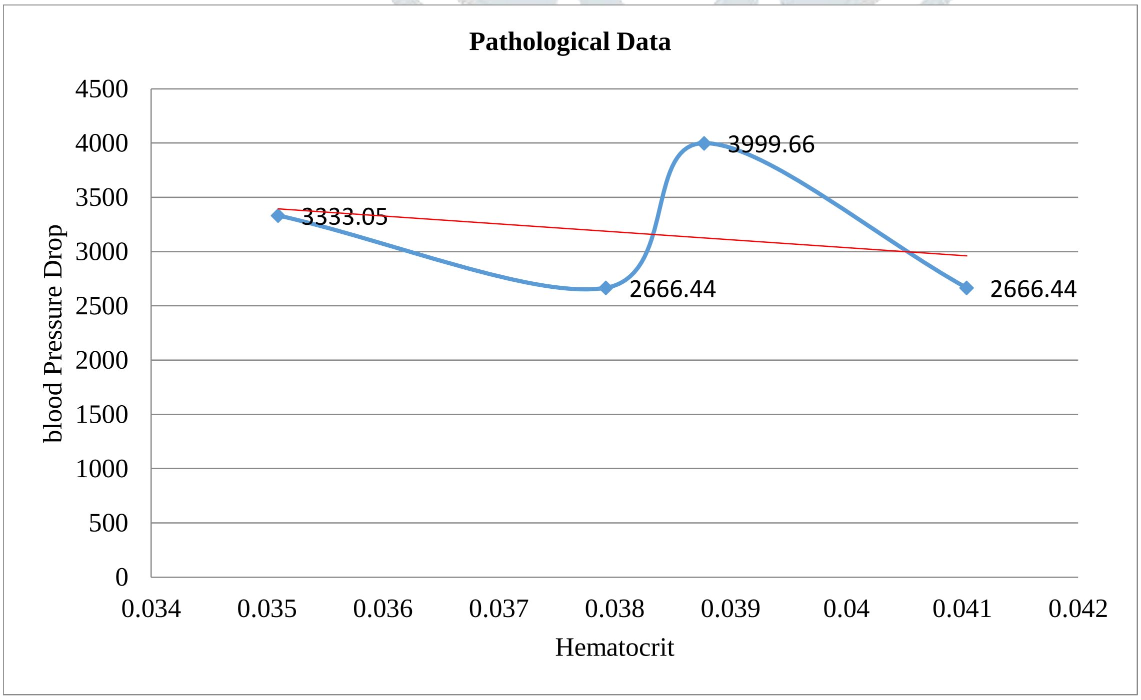
$$\Delta P = 1.907678565H + 0.00299667$$

4.2. Table for Hematocrit v/s Pressure drop

Hematocrit	0.035095	0.037925	0.038774	0.041038
Blood Pressure drop	0.069946649	0.07534538	0.076964999	0.081283983

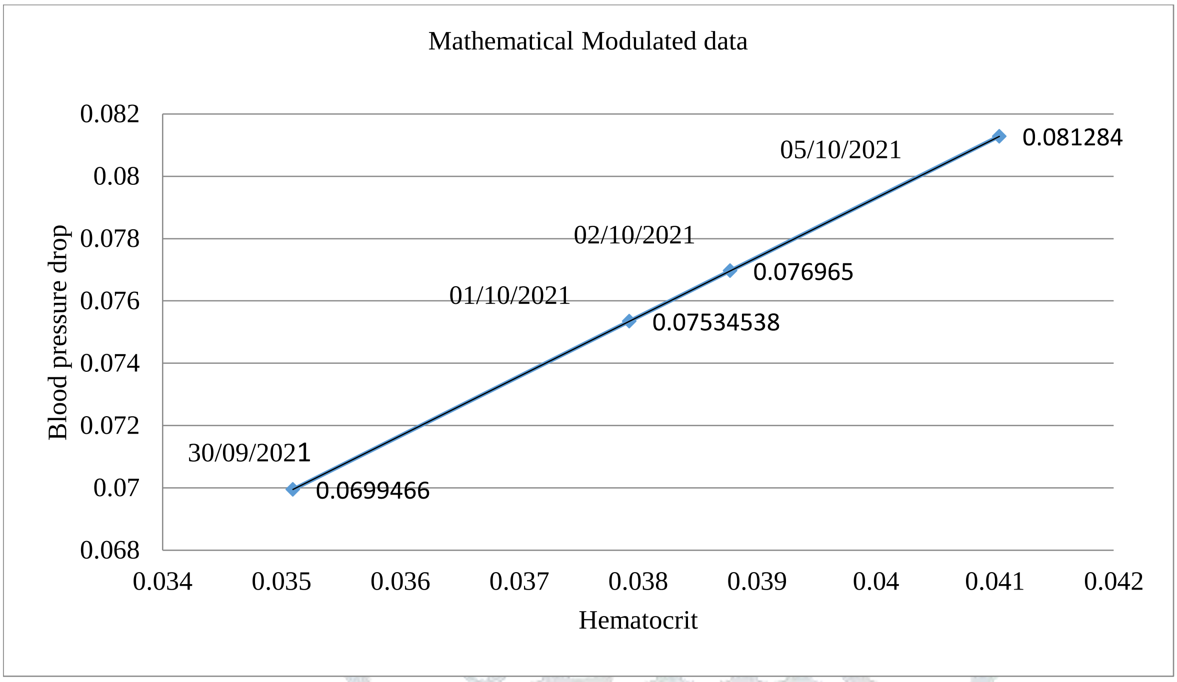
Table 4.2

(a). Graphical Presentation of Pathological data:



Graph 1: Relation between Hematocrit V/S Pathological Blood Pressure Drop

(b): Graphical Presentation of Mathematical Modulate Data:



Graph 1: Relation between Hematocrit V/S Mathematical Modulated Blood Pressure Drop.

5. Conclusion:

According to this study of the graph between hematocrit and blood pressure drop in suffering silent ischemia patient we have concluded that when hematocrit increased then the blood pressure drop is also increased and shows a linear graph.

6. Acknowledgment:

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