



MIXED FINITE ELEMENT ANALYSIS OF FUNCTIONALLY GRADED BEAMS USING MATLAB COMPUTER PROGRAMMING

¹Rasesh Patil, ²Sanjay Denge, ³Girish Sawai

¹PG Student, ²Professor, ³Head of Department,

¹²³Department of Civil Engineering

¹²³VM Institute of Engineering & Technology Nagpur, Maharashtra, India.

ABSTRACT

This study is carried out for the finite element analysis of Functionally Graded beams using a software Matlab. The functionally graded beam is a beam having varying Young's modulus along its depth and we will be using Matlab computer programming to analyse the behaviour of the beam in certain stress conditions. We will be achieving this by calculating the present analysis by our programming and comparing the result with available solutions of The Mixed semi analytical solution, 2D exact elasticity solution, 1D analytical FOST solution and HOSNT beam solutions. The error in the solutions will be mentioned along with our present analysis as the result and accuracy of our analysis. The numerical example which will be discussed will show the viability of the Mixed Finite Element approach to tackle such problems also the results derived have shown good correlations with solutions.

1. Introduction

The demand on structures to carry increased magnitude of loads and maintain functionality under conditions that are detrimental to a conventional material's properties, leads to the need for development of a material that has low specific weight, significantly higher magnitudes of resistance to fracture, strength, durability, better young's modulus and ability to retain these properties under very high working temperatures. A closer look suggests that the potential of most alloys, which form the main category of Structural materials have been taken to the limit, this seems to point in the direction of ceramics. But ceramics present problems of unsatisfactory fracture toughness for most implementations. Glass Fiber Reinforced Polymers and Carbon Fiber Reinforced Polymers have been a good breakthrough technology. However, they are highly directional, and the anisotropy must be controlled very carefully. To overcome this, Composite laminates were developed, which were developed specifically to provide desired stiffness properties in longitudinal and lateral axis directions through a mix of longitudinal and lateral laminates oriented a specific sequence. The orientation of each layer is arbitrary, and the layup sequence is tailored to achieve the properties desired of the laminate. Depending upon the orientation of fibers, the laminates are classified as unidirectional, cross-ply, angle-ply, hybrid, etc. Though the strain variation across the depth is presumably linear, the sudden difference in material properties across the interfaces of the laminae leads to offsetting stress distribution, causing high inter-laminar stresses. In the absence of any graded material at the interface, which creates a significant chance of delamination. To overcome disadvantages of issues.

Functionally Graded Beam- Functionally Graded Materials are idealized as a gradual variation in material composition and structure, mainly in thickness direction, which leads to a gradual change in the properties of the material. By gradually varying the volume fraction of constituent materials, the FGM exhibits a smooth and continuous property change from one surface to another. This results in an un-offsetting variation in stresses across the section, which considerably reduces the inter-laminar stresses and the risk of de-lamination.

Mixed Finite Element Approach- It is a numerical analysis method which is a hybrid finite element method, wherein extra independent variables are introduced as nodal variables. The extra independent variables are constrained by using Lagrange multipliers. In structural analysis, when conventional finite element methods are utilized for problem solving, although they provide satisfactory global and local values of deflections and flexural stresses and the continuity of displacement fields through thickness are satisfied, they create problems regarding continuity of transverse stress components at interfaces through the thickness, which becomes difficult to enforce.

2. Objective of work

- To study a Mixed Finite Element analysis formulation for Functionally Graded Beam Element with Exponential law variation.
- To validate present formulation by numerical investigation and comparing the results with existing solutions

3. Scope of the work

- Development of a computer program for obtaining numerical results.
- To validate the developed computer program by comparing the numerical results with solutions available in existing literature.

4. Literature Review

Vast amount of research has been done on Functionally graded beams on various theories and the behavior of beams under various support conditions have been analyzed. Following topics have been covered and are discussed below:

Solutions Based on Euler Bernoulli Beam Theory:

The elementary theory of beam bending introduced by Euler and Bernoulli is the simplest theory of beam bending which neglects the effects of the shear deformation. It is based on the assumption that the plane section which is normal to the neutral axis before bending remains plane and normal to the neutral axis even after deformation of the beam. That means shear stresses are not considered in the hypothesis.

Aldousari (2017) developed two symmetric and anti-symmetric functions and compare their effects on the static deflection and bending stresses with classical power-law distribution. The proposed distributions are a symmetric power-law and a sigmoid function which is anti-symmetric. Kinematic relation of Euler–Bernoulli beam is assumed, and virtual work is proposed to derive the equilibrium equations. A finite element model is proposed to form stiffness matrix and force vector and then solve the problem numerically.

Solutions based on Higher Order Shear Deformation Beam Theory:

Kant *et al.* (1988) first developed a HOSBT beam model using linear variation of transverse normal strain and parabolic variation of transverse shearing strain across the depth of the beam. Material constitutive law is used to invoke effects of transverse normal stresses and transverse shear stresses, thus warping of transverse normal cross-section is taken into account in the mathematical model.

Reddy *et al.* (1997) have developed relationships between the deflections and rotations, bending moments and shear forces of a third-order beam theory, and those of the Euler-Bernoulli theory and the Timoshenko beam theory. So, if one has the solutions for beam having a given loading and boundary conditions based on EBT, then the relation by developed by Reddy can be utilized to find the solution for the beam for the same conditions based on Third Order Beam Theory. The results for these beam theories are studied for the FGM beam by Karamanli (2016). The investigations are done for electrostatic deformations of the FGM beams for various boundary conditions using a Symmetric Smoothed Particle Hydrodynamics (SSPH) method. The comparison of various beam theories using SSPH method was done for the first time for beams constituting two different materials. Thai (2012) developed higher order shear deformation theories for Bending and free vibration. Here the shear stresses at the boundaries were zero, so there was no need for developing the shear correction factors. Results were verified against analytical solutions.

Tharaknath (2014) studied the static analysis for a simply supported Functionally Graded beam subjected to a UDL through Ritz method and his beam theory was based on TBT and HOST. Transverse deflections and axial displacements and rotations of beams cross sections were idealized as trigonometric functions.

Meghare and Jadhao (2015) applied the Refined Theory for flexural analysis of FGM beams. The in-plane displacement field uses sinusoidal function in terms of thickness coordinates thus including the shear deformation effect. The functionally graded beam is analyzed for uniformly distributed load and linearly varying load under simply supported condition

Solutions based on Finite Element Method:

Chakraborty and Reddy (2003) developed a new beam element to study the thermoelastic behavior of functionally graded beam structures. The element is based on the FOSDT and it takes into account varying elastic and thermal properties along its thickness. Exact solutions of static part of the governing differential equations is used to develop interpolating polynomials for the element formulation. Thus, the developed beam element doesn't have shear locking and the stiffness matrix is super-convergent property. Sayyad and Ghugal (2018) discussed different techniques which are used to predict the effective properties at point, material idealization, and methods to solve single layered Functionally Graded Beams. They also found that for beams that have a steel rich bottom, having variation in property based on power law are relatively more stable when compared to beams which have exponential law variation in properties.

Chakraborty *et al.* (2003) developed a finite element based on TBT in which they derived shape functions from the general exact solution of the static governing equations. Both Polynomial and Exponential variation of elastic and thermal properties along thickness were considered. Li (2008) developed a unified approach to the formulation of the EBT and TBT. Kadoli *et al.* (2008) developed and FEM based on a third-order approximation of the axial displacement and constant transverse displacement for ceramic and metal Functionally Graded Beams. Power law gradation was used. A discrete layer approach was chosen to take into account material gradation. Functionally Graded Materials have been considered within the CUF for higher-order beam modeling by Giunta *et al.* (2010)

Analytical Solutions:

Availability of exact elasticity solutions is important as they serve as benchmarks for comparison against solutions from beam theories based on approximations. However, there are rare contributions towards exact elasticity solutions. Sankar (2000) obtained an elasticity solution for an FG beams having sinusoidal transverse loading. He observed that beam theory is valid for slender long beams with transverse loading slowly varying. He also concluded that stress concentrations when compared to a conventional homogenous beam were found that to be in more numbers when the beam is loaded on the stiffer side and in lesser numbers when it is loaded on the softer side.

Pendhari *et al* (2010) obtained a solution for FG narrow beams under plane stress conditions by using a mixed semi analytical model developed by Kant. Their mathematical model consist of a two point boundary value problem governed by set of coupled FODEs in the thickness direction by assuming the primary variables in the form of trigonometric functions along the longitudinal direction of beam which satisfies the simply supported conditions exactly. Highlight of the mixed semi analytical formulation is that the governing differential equation system is not transformed into an algebraic equation system, thus the intrinsic behavior of the physical system is retained to a greater degree of accuracy.

Chu (2015) obtained elasticity solution of beams made of FG materials under tension and bending. Based on a 2D theory of elasticity, a governing equation is derived by means of the Airy stress function method together with the strain compatibility equation.

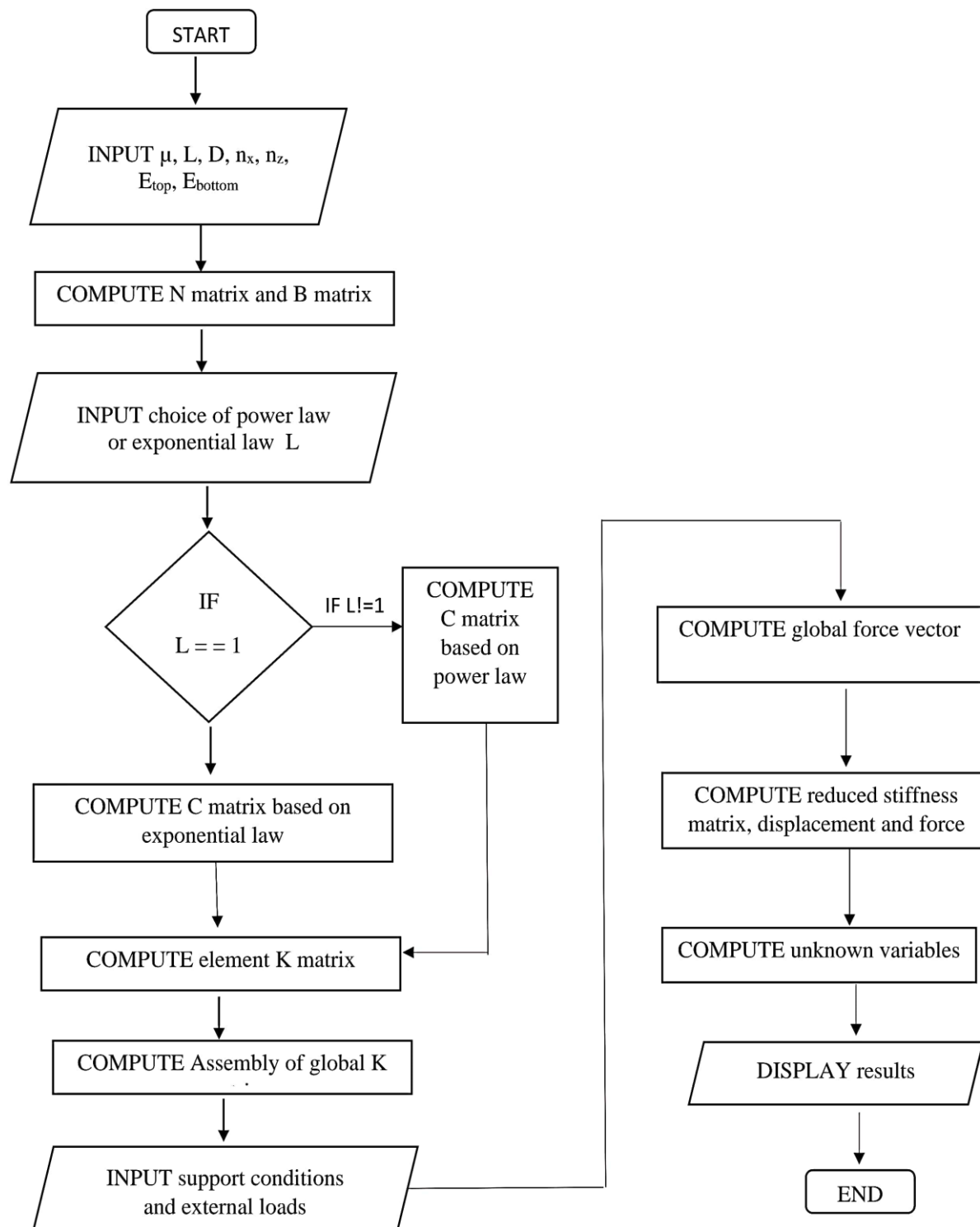
Mixed Finite Element Solutions:

The technique of Mixed Finite Element Method utilizes the introduction of independent variables as degrees of freedom. These are invoked by using fundamental elasticity equations.

Desai *et al.* (2002) developed a 6 noded 2-D model and used it to analyze laminated composite beams on the principle of minimum potential energy. Each node had 4 degrees of freedom, 2 displacements and 2 transverse stress components.

5. Program Flow Chart

Following is the flow chart used for the programming done in software MATLAB. This software is developed to study and analyse the behaviour of functionally graded beams under various transverse loading conditions and with different boundary conditions.

Program Flow Chart:**6. Numerical Study and Result**

The accuracy of results obtained from the computer program based on the current formulation is validated and compared with results in available literature.

The percentage error in results is calculated as,

$$\text{Percentage error} = \left[\frac{\text{Present value} - \text{Elasticity value}}{\text{Elasticity value}} \right] \times 100$$

The following set of material properties are used:

Material set 1 (Sankar, 2001)

$E_o = 1$ GPa

$Eh / E_o = 5, 10, 20$ and 40

$\nu=0.3$

Example 1

A simply supported FG beam loaded with uniformly distributed load is considered here. Material set 1 is used. The ratio of Young's modulus of top fibre to bottom fibre (λ) is taken as 5. The normalized transverse displacement ($\bar{w}$), inplane displacement ($\bar{u}$), normalized inplane normal stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) for different aspect ratios are presented in Table 1.1. The Mixed semi analytical solution, 2D exact elasticity solution, 1D analytical FOST solution and HOSNT beam solutions are used for comparison. Through thickness variations of inplane displacement ($\bar{u}$), transverse displacements ($\bar{w}$), inplane normal stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) for an aspect ratio of 5, 10 are shown in Figure 1.1., Figure 1.2. respectively

Results and Comparison:

Table 1.1: Normalized transverse displacement ($\bar{w}$), Normalized shear transverse stresses ($\bar{\tau}_{xz}$) and Normalized inplane stresses ($\bar{\sigma}_x$) of FG beam under uniformly distributed transverse load.

Variables	L/h	Present Analysis	Elasticity Solution	Semi analytical Solution	HOSNT
$\bar{w}\left(\frac{L}{2}, h\right)$	5	38.574	38.6454	38.6405	38.6315
		(-0.171)		(-.001)	(-.025)
	10	36.5502	36.3563	36.3563	36.3540
		(1.225)		(.000)	(-.006)
$\bar{u}(0,0)$	5	5.12111	3.853	3.3866	3.853
		(-32.92)		(0.0206)	(.000)
	10	31.6686	29.79	29.71	29.662
		(6.64)		(-0.269)	(-0.431)
$\bar{\sigma}_x\left(\frac{L}{2}, h\right)$	5	1.212	1.2938	1.2902	1.2927
		(-6.32)		(.148)	(.202)
	10	1.2001	1.2746	1.2747	1.2753
		(-.584)		(.008)	(.055)

Note: Values reported in the table are at the center of beam ($L/2$) and at a top surface (h)

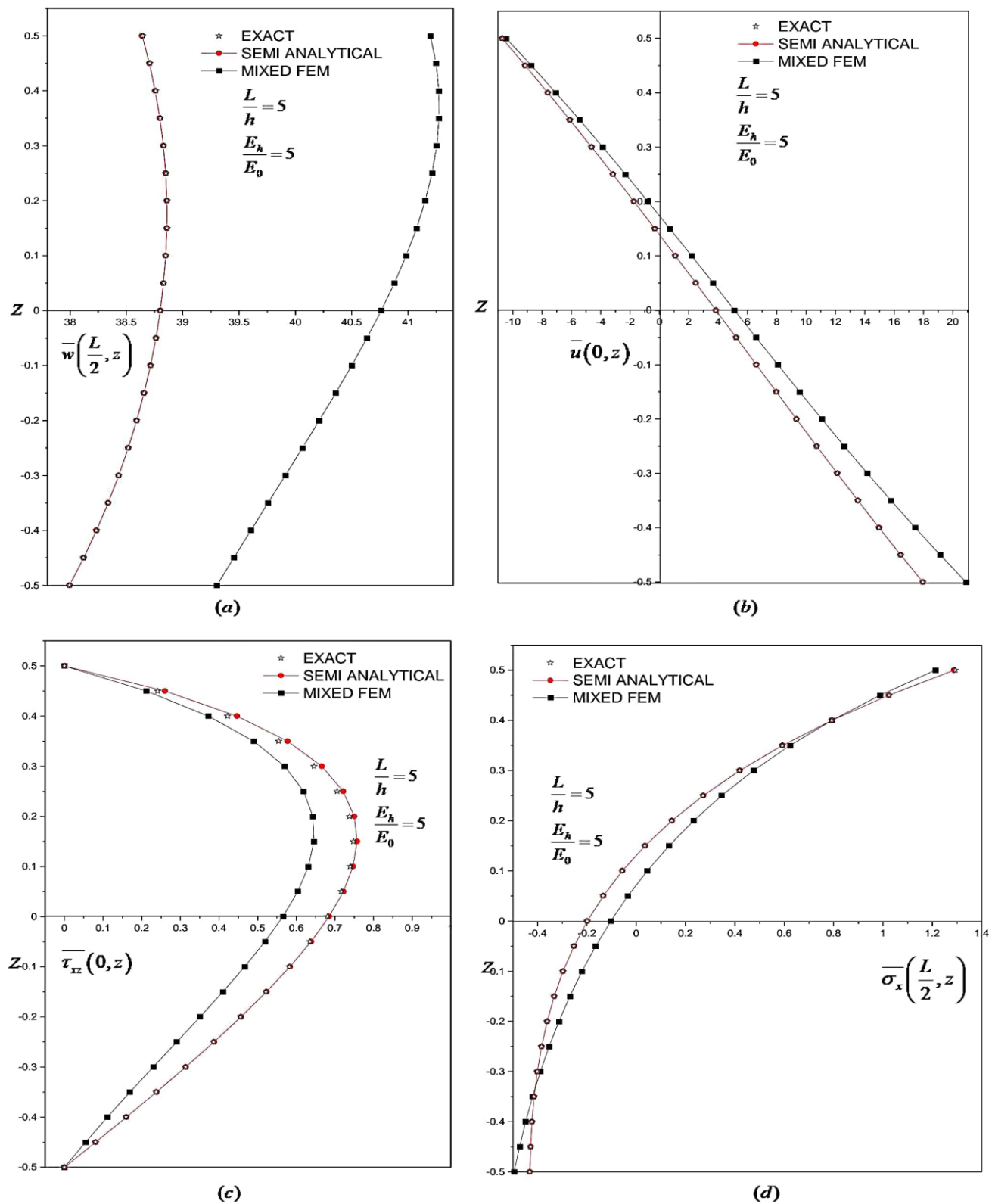


Figure 1.1: Through thickness variation of (a) transverse displacement $\bar{w}$, (b) inplane displacement $\bar{u}$, (c) transverse shear stress $\bar{\tau}_{xz}$ and (d) inplane normal stress $\bar{\sigma}_x$ for simply supported FG beam under uniformly distributed load for aspect ratio 5.

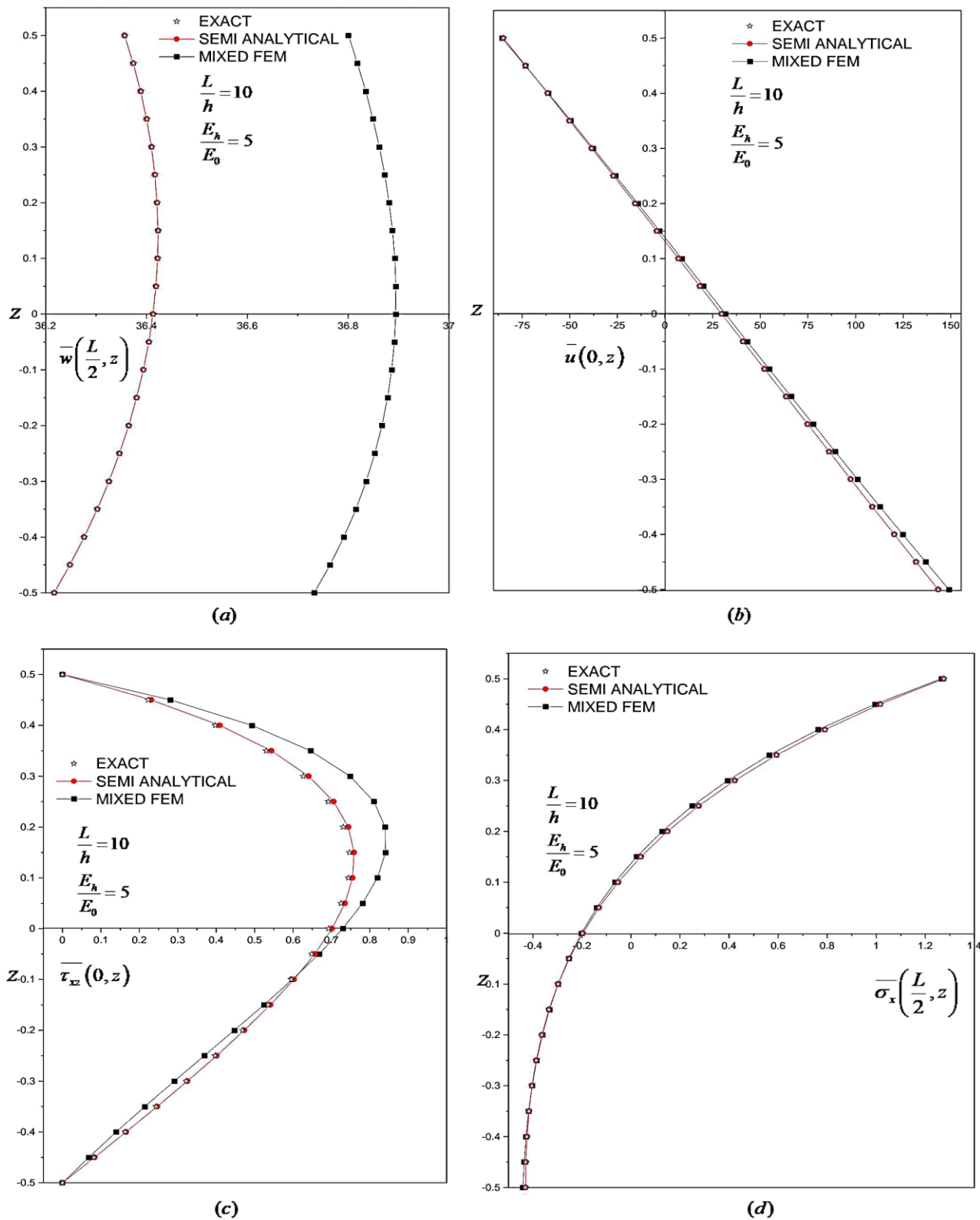


Figure 1.2: Through thickness variation of (a) transverse displacement $\bar{w}$, (b) inplane displacement $\bar{u}$, (c) transverse shear stress $\bar{\tau}_{xz}$ and (d) inplane normal stress $\bar{\sigma}_x$ for simply supported FG beam under uniformly distributed load for aspect ratio 10.

7. Concluding Remarks

Analysis of functionally graded beams have been thoroughly done with the help of various theories and the program coding done in Matlab. The whole point of utilizing the mixed finite element approach was to ensure continuity of both displacement and transverse stress field across the thickness. This feature of their formulation has not been observed in any other mixed finite element formulation and thus is novel approach. The numerical example solved show the viability of the mixed finite element approach to tackle such problems, the results have shown a good correlation with well-established solutions. The program written and executed successfully using Matlab platform and the results were displayed in comparison with existing literature.

8. References

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